

Compressible hydromagnetic nonlinearities in the predecoupling plasma

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Abstract

The adiabatic inhomogeneities of the scalar curvature lead to a compressible flow affecting the dynamics of the hydromagnetic nonlinearities. The influence of the plasma on the evolution of a putative magnetic field is explored with the aim of obtaining an effective description valid for sufficiently large scales. The bulk velocity of the plasma, computed in the framework of the Λ CDM scenario, feeds back into the evolution of the magnetic power spectra leading to a (nonlocal) master equation valid in Fourier space and similar to the ones discussed in the context of wave turbulence. Conversely, in physical space, the magnetic power spectra obey a Schrödinger-like equation whose effective potential depends on the large-scale curvature perturbations. Explicit solutions are presented both in physical space and in Fourier space. It is argued that curvature inhomogeneities, compatible with the WMAP 7yr data, shift to lower wavenumbers the magnetic diffusivity scale.

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1 Introduction

The evolution equations of the magnetic power spectra in a conducting fluid have been the subject of extended studies with particular attention to the regime of high Reynolds numbers (see, e.g. [1, 2, 3, 4]). In standard treatments the flow is often passive (in the sense that it is given as an external input of the problem), it is isotropic and it is assumed to be incompressible in various situations ranging from the usual one-fluid description provided by magnetohydrodynamics (see, e.g. [4]) to the applications of gyrotropic turbulence [3]. Exceptions to the previous statement are, for instance, the compressible turbulence [5] or the hydromagnetic evolution in the presence of acoustic disturbances.

Prior to photon decoupling, the physical properties of the primeval plasma can be directly scrutinized by means of the measurements of the Cosmic Microwave Background (CMB) temperature and polarization anisotropies. These observations show, with an accuracy which is sufficient for the present considerations, the predominance of the adiabatic curvature perturbations over any other (possibly subleading) entropic contribution [6, 7] (see also [8] for a recent review). Indeed, the results of the WMAP observations (see, e.g. the WMAP 7yr data release [9, 10, 11]) imply that the position of the first acoustic peak of the temperature autocorrelations and the position of the first anticorrelation peak in the temperature-polarization power spectra are in a fixed ratio (i.e. approximately 3/4). Such a ratio can be easily derived by assuming a dominant adiabatic component of curvature inhomogeneities in the initial conditions of the Einstein-Boltzmann hierarchy. Tensor and vector modes play a negligible role (see [12, 13] and references therein) but are anyway absent in the vanilla Λ CDM scenario (where Λ stands for the dark energy component and CDM for the cold dark matter component). The adiabatic nature of the large-scale curvature perturbations determines also the statistical properties of the fluid flow: the baryon-photon and the cold dark matter velocities are Gaussian, isotropic and irrotational. The bulk velocity of the plasma (i.e. \vec{v}_b in what follows) coincides, at early times, with the baryon velocity [14] (see also [15]). When photons and baryons are tightly coupled the photon-baryon flow is also compressible and the typical amplitude of the corresponding velocity field is determined by the large-scale curvature perturbations. Finally the correlation functions of the bulk velocity of the plasma is exponentially suppressed for large times beyond a typical time scale, denoted by τ_d and related to the diffusive (or Silk) damping.

The main motivation of this paper can be summarized as follows. Prior to decoupling the Prandtl number (i.e. $Pr_{\text{magn}} = R_{\text{magn}}/R_{\text{kin}}$) is $Pr_{\text{magn}} = \mathcal{O}(10^{20})$ with $R_{\text{kin}} < 1$ and $R_{\text{magn}} \gg 1$. In such a system the fluid flow is computable (for instance in one of the popular versions of the Λ CDM scenario) and, in some sense, even accessible experimentally. Consequently, a natural question to ask is whether it is possible to account for the effect of the large-scale flow on the evolution of the magnetic power spectra. This problem is formally analog to (but physically very different from) some of the themes mentioned in the first paragraph of this introductory section. In the recent past close attention has been

paid to the analysis of the effects of large-scale magnetic fields on the evolution of curvature perturbations and on the CMB anisotropies (see, e.g., [14, 16] and references therein). In this analysis the complementary (and to some extent inverse) point of view will be taken: given the properties of the large-scale flow in the framework of the Λ CDM paradigm we ought to know how is the evolution of a putative magnetic field modified, in particular for large-scales.

A similar problem arises in the statistical treatment of conducting fluids [1, 2, 3] where an interesting approach has been pioneered by Kazantsev [17], Kraichnan and Nagarajan [18] (see also [19, 20]) and Vainshtein [21]. The common aspect of the approaches of [17, 18, 21] is that the flow is assumed Gaussian, isotropic and incompressible: these are also the three main physical analogies with the situation addressed in this paper.

In spite of these relevant analogies there are also two important differences. The first one is that the fluctuations of the geometry (as well as the effects related to the expansion of the background) do not play any role in the hydromagnetic treatment of conducting fluids; conversely, prior to decoupling, the large-scale flow is exactly determined by the fluctuations of the spatial curvature. In this sense the flow discussed in the present analysis is not passive but rather computed from the large-scale curvature perturbations. The second physical difference concerns the hierarchy between the Reynolds numbers which are simultaneously high (i.e. $R_{\text{kin}} \gg 1$ and $R_{\text{magn}} \gg 1$) in the standard hydromagnetic treatments² [17] (see also [18, 21]). This is not the case after electron-positron annihilation and the smallness of R_{kin} is partly related to the properties of the bulk flow whose evolution is described within the same perturbative treatment used for the large-scale curvature fluctuations.

The layout of the present paper is the following. In section 2 the set of governing equations will be introduced in its general form. In section 3 the Λ CDM fluid flow will be fed back into the evolution of the magnetic fields leading to an explicit integrodifferential equation valid for a compressible flow. The evolution of the magnetic power spectra is derived both in Fourier space and in physical space (section 4). Explicit solutions are derived in section 5. Section 6 contains the concluding remarks. In the appendixes useful technical results have been collected for the interested readers.

2 Governing equations

The hydromagnetic equations in the presence of the fluctuations of the geometry have been deduced in different gauges (see, e.g. [14, 16] and references therein). Here the essentials of the curved space description of the predecoupling plasma will be briefly reviewed. The approach of this paper is, in some sense, inverse of the analysis of [14, 16]: we are here concerned

² In spite of the fact that the three approaches of [17, 18, 21] are conceptually related, the analysis of Kazantsev [17] seems more appropriate for the present ends and will therefore constitute the basis for the generalizations developed in the present paper.

with the problem of deducing an averaged description for the evolution of the magnetic power spectra when the flow is determined in the framework of the Λ CDM paradigm.

2.1 Curvature perturbations

In a conformally flat background metric $\bar{g}_{\mu\nu} = a^2(\tau)\eta_{\mu\nu}$ the relativistic fluctuations of the geometry are given by

$$\delta_s g_{00}(\vec{x}, \tau) = 2a^2(\tau)\phi(\vec{x}, \tau), \quad \delta_s g_{ij}(\vec{x}, \tau) = 2a^2(\tau)\psi(\vec{x}, \tau)\delta_{ij}, \quad (2.1)$$

in the longitudinal coordinate system. In Eq. (2.1) δ_s denotes a metric perturbation which preserves the scalar nature of the fluctuation since, in the Λ CDM paradigm, the dominant source of inhomogeneity comes from the scalar modes of the geometry while the tensor and the vector inhomogeneities are absent. The scalar inhomogeneities are customarily parametrized either in terms of the curvature fluctuations on comoving orthogonal hypersurfaces (conventionally denoted with \mathcal{R}) or in terms of the curvature perturbations on uniform density hypersurfaces (conventionally denoted by ζ). Both \mathcal{R} and ζ can be defined in terms of the variables ϕ and ψ appearing in Eq. (2.1):

$$\mathcal{R} = -\psi - \frac{\mathcal{H}(\mathcal{H}\phi + \partial_\tau\psi)}{\mathcal{H}^2 - \partial_\tau\mathcal{H}}, \quad \zeta = \mathcal{R} + \frac{\nabla^2\psi}{3(\mathcal{H}^2 - \partial_\tau\mathcal{H})}, \quad (2.2)$$

where $\mathcal{H} = aH$ and H is the usual Hubble rate. The variables \mathcal{R} and ζ are gauge-invariant and the second relation of Eq. (2.2) stems from the Hamiltonian constraint written in the longitudinal gauge of Eq. (2.1). For a more thorough definition of large-scale curvature perturbations in different coordinate systems see [15, 23, 24] and references therein. The large-scale curvature perturbations are then assigned in terms of the two-point function of \mathcal{R}_* , in Fourier space and prior to matter-radiation equality where curvature perturbations are approximately constant, at least in the case of the Λ CDM paradigm:

$$\langle \mathcal{R}_*(\vec{k}) \mathcal{R}_*(\vec{p}) \rangle = \frac{2\pi^2}{k^3} \mathcal{P}_{\mathcal{R}}(k) \delta^{(3)}(\vec{k} + \vec{p}), \quad \mathcal{P}_{\mathcal{R}}(k) = \mathcal{A}_{\mathcal{R}} \left(\frac{k}{k_p} \right)^{n_s-1}. \quad (2.3)$$

In Eq. (2.3) $\mathcal{A}_{\mathcal{R}} = 2.43 \times 10^{-9}$ is the amplitude of the power spectrum at the pivot scale $k_p = 0.002 \text{ Mpc}^{-1}$ and $n_s = 0.963$ is, by definition, the scalar spectral index. The numerical values of $\mathcal{A}_{\mathcal{R}}$ and n_s are determined from the WMAP 7yr data analyzed in the light of the vanilla Λ CDM scenario; the values of the remaining parameters are given by [9, 10, 11]:

$$(\Omega_{b0}, \Omega_{c0}, \Omega_{de0}, h_0, n_s, \epsilon_{re}) \equiv (0.0449, 0.222, 0.734, 0.710, 0.963, 0.088), \quad (2.4)$$

where, as usual, Ω_{x0} denotes the present critical fraction of the species x ; h_0 is the Hubble constant in units of 100 km/sec/Mpc and ϵ_{re} is the optical depth to reionization.

2.2 Two-fluid equations

The comoving electromagnetic fields and the comoving concentrations of electrons and ions are given by

$$\begin{aligned}\vec{E}(\vec{x}, \tau) &= a^2(\tau) \vec{\mathcal{E}}(\vec{x}, \tau), & \vec{B}(\vec{x}, \tau) &= a^2(\tau) \vec{\mathcal{B}}(\vec{x}, \tau), \\ n_i(\vec{x}, \tau) &= a^3(\tau) \tilde{n}_i(\vec{x}, \tau), & n_e(\vec{x}, \tau) &= a^3(\tau) \tilde{n}_e(\vec{x}, \tau).\end{aligned}\quad (2.5)$$

Thus Maxwell's equations read

$$\vec{\nabla} \cdot \vec{E} = 4\pi e(n_i - n_e), \quad \vec{\nabla} \cdot \vec{B} = 0, \quad (2.6)$$

$$\vec{\nabla} \times \vec{E} = -\partial_\tau \vec{B}, \quad \vec{\nabla} \times \vec{B} = 4\pi e(n_i \vec{v}_i - n_e \vec{v}_e) + \partial_\tau \vec{E}. \quad (2.7)$$

The velocities of the electrons, ions and photons obey, respectively, the following set of coupled equations:

$$\partial_\tau \vec{v}_e + \mathcal{H} \vec{v}_e = -\frac{en_e}{\rho_e a^4} [\vec{E} + \vec{v}_e \times \vec{B}] - \vec{\nabla} \phi + \frac{4}{3} \frac{\rho_\gamma}{\rho_e} a \Gamma_{\gamma e} (\vec{v}_\gamma - \vec{v}_e) + a \Gamma_{ei} (\vec{v}_i - \vec{v}_e), \quad (2.8)$$

$$\partial_\tau \vec{v}_i + \mathcal{H} \vec{v}_i = \frac{en_i}{\rho_i a^4} [\vec{E} + \vec{v}_i \times \vec{B}] - \vec{\nabla} \phi + \frac{4}{3} \frac{\rho_\gamma}{\rho_i} a \Gamma_{\gamma i} (\vec{v}_\gamma - \vec{v}_i) + a \Gamma_{ei} \frac{\rho_e}{\rho_i} (\vec{v}_e - \vec{v}_i), \quad (2.9)$$

$$\partial_\tau \vec{v}_\gamma = -\frac{1}{4} \vec{\nabla} \delta_\gamma - \vec{\nabla} \phi + a \Gamma_{\gamma i} (\vec{v}_i - \vec{v}_\gamma) + a \Gamma_{\gamma e} (\vec{v}_e - \vec{v}_\gamma). \quad (2.10)$$

In Eqs. (2.8)–(2.10) the relativistic fluctuations of the geometry are included from the very beginning in terms of the longitudinal gauge variables of Eq. (2.1); the electron-photon, electron-ion and ion-photon rates of momentum exchange appearing in Eqs. (2.8)–(2.10) are given by³:

$$\Gamma_{\gamma e} = \tilde{n}_e \sigma_{e\gamma}, \quad \Gamma_{\gamma i} = \tilde{n}_i \sigma_{i\gamma}, \quad \sigma_{e\gamma} = \frac{8}{3} \pi \left(\frac{e^2}{m_e} \right)^2, \quad \sigma_{i\gamma} = \frac{8}{3} \pi \left(\frac{e^2}{m_i} \right)^2, \quad (2.11)$$

$$\Gamma_{ei} = \tilde{n}_e \sqrt{\frac{T}{m_e}} \sigma_{ei} = \Gamma_{ie}, \quad \sigma_{ei} = \frac{e^4}{T^2} \ln \Lambda_C, \quad \Lambda_C = \frac{3}{2e^3} \sqrt{\frac{T^3}{\tilde{n}_e \pi}}. \quad (2.12)$$

In Eq. (2.11) and (2.12), T and \tilde{n} are, respectively, physical temperatures and physical concentrations⁴.

2.3 The problem of the closure

The two global variables entering the one-fluid description are the center of mass velocity of the electron ion system and the total current, i.e.

$$\vec{v}_b = \frac{m_e \vec{v}_e + m_i \vec{v}_i}{m_e + m_i}, \quad \vec{J} = n_i \vec{v}_i - n_e \vec{v}_e. \quad (2.13)$$

³Note that Λ_C is the Coulomb logarithm [25, 29].

⁴If the rates and the cross sections are expressed in terms of comoving temperatures $\bar{T} = aT$ and comoving concentrations $n = a^3 \tilde{n}$ the corresponding rates will inherit a scale factor for each mass. For instance $a \Gamma_{ei}$ becomes $n_e \sqrt{\bar{T}/(m_e a)} (e^4/\bar{T}^2) \ln \Lambda_C$, if comoving temperature and concentrations are used.

The one-fluid equations supplemented by the incompressible closure are often used as a starting point for the analysis especially in connection with the potential emergence of interesting scaling laws (see, e.g. [31]). The bulk velocity of the plasma coincides, in the latter case, with \vec{v}_b and the incompressibility condition dictates $\vec{\nabla} \cdot \vec{v}_b = 0$. Having said this, the reduction from the two-fluid to the one-fluid variables is essential, in the present context, exactly because $\vec{\nabla} \cdot \vec{v}_b \neq 0$ and the velocity correlators depend on the density fluctuations of the plasma. The difference of Eqs. (2.8) and (2.9) leads to the evolution equation of the total current. Since the rate of Coulomb scattering is much larger than the conformal time derivatives of the current, the Ohm equation can be approximated by the standard form of the Ohm law (see fifth article of Ref. [14]):

$$\vec{J} = \sigma \left(\vec{E} + \vec{v}_b \times \vec{B} + \frac{\vec{\nabla} p_e}{en_0} - \frac{\vec{J} \times \vec{B}}{en_0} \right), \quad (2.14)$$

where $n_0 = a^3 \tilde{n}_{ei}$ denotes the common value of the (common) electron-ion concentration; σ is the conductivity of the predecoupling plasma. After electron-positron annihilation $\sigma(\bar{T})$ can be estimated as

$$\sigma(\bar{T}) = \sigma_1 \frac{\bar{T}}{\alpha_{em}} \sqrt{\frac{\bar{T}}{m_e a \ln \Lambda_C}}, \quad \Lambda_C(\bar{T}) = \frac{3}{2e^3} \left(\frac{\bar{T}^3}{\pi n_e} \right)^{1/2} = 1.105 \times 10^8 \left(\frac{\omega_{b0}}{0.02258} \right)^{-1/2}, \quad (2.15)$$

where $\sigma_1 = 9/(8\pi\sqrt{3})$ depends on the way multiple scattering is estimated.

The sum of Eqs. (2.8) and (2.9) leads to the evolution equation of the bulk velocity of the plasma \vec{v}_b which is directly coupled to the photon velocity and to the density contrasts of photons (i.e. δ_γ) and baryons (i.e. δ_b):

$$\partial_\tau \vec{v}_b + \mathcal{H} \vec{v}_b = \frac{\vec{J} \times \vec{B}}{a^4 \rho_b} - \vec{\nabla} \phi + \nu_b \nabla^2 \vec{v}_b + \frac{4\rho_\gamma}{3\rho_b} \epsilon' (\vec{v}_\gamma - \vec{v}_b), \quad (2.16)$$

$$\partial_\tau \vec{v}_\gamma = -\frac{1}{4} \vec{\nabla} \delta_\gamma - \vec{\nabla} \phi + \nu_\gamma \nabla^2 \vec{v}_\gamma + \epsilon' (\vec{v}_b - \vec{v}_\gamma), \quad (2.17)$$

$$\partial_\tau \delta_b = 3\partial_\tau \psi - \vec{\nabla} \cdot \vec{v}_b + \frac{\vec{J} \cdot \vec{E}}{\rho_b a^4}, \quad (2.18)$$

$$\partial_\tau \delta_\gamma = 4\partial_\tau \psi - \frac{4}{3} \vec{\nabla} \cdot \vec{v}_\gamma, \quad (2.19)$$

where ν_b and ν_γ denote the thermal diffusivity coefficients and $\epsilon' = \tilde{n}_e \sigma_{\gamma e} a$ is the differential optical depth of electron-photon scattering. Equations (2.16)–(2.19) are all consistently written in the longitudinal coordinate system of Eq. (2.1).

Dropping the third and fourth terms in Eq. (2.14) (i.e. the thermoelectric and Hall terms) the evolution equation of the magnetic fields reads, in the one-fluid approximation,

$$\partial_\tau \vec{B} = \vec{\nabla} \times (\vec{v}_b \times \vec{B}) + \lambda \nabla^2 \vec{B}, \quad \lambda = \frac{1}{4\pi\sigma}. \quad (2.20)$$

If the Hall and the thermoelectric terms are kept in Eq. (2.14), Eq. (2.20) becomes⁵

$$\partial_\tau \vec{B} = \vec{\nabla} \times (\vec{v}_b \times \vec{B}) + \frac{1}{4\pi\sigma} \nabla^2 \vec{B} + \vec{\nabla} \times \left(\frac{\vec{\nabla} p_e}{en_0} \right) - \frac{1}{4\pi en_0} \vec{\nabla} \times [(\vec{\nabla} \times \vec{B}) \times \vec{B}]. \quad (2.21)$$

As in [17] we shall assume that the magnetic field is sufficiently small so that the Hall term is negligible (see, however, [28]). We shall also assume that there is no particular alignment between the gradient of the concentration and of the electron pressure so that the thermoelectric term is also negligible.

The standard incompressible closure stipulates that the velocity field is solenoidal but this is not the situation described by Eqs. (2.16)–(2.20) where, in general, $\vec{\nabla} \cdot \vec{v}_\gamma \neq 0$ and $\vec{\nabla} \cdot \vec{v}_b \neq 0$. Prior to photon decoupling the electron-photon scattering rate drives the baryon and photon velocities to a common value which will be denoted by $\vec{v}_{\gamma b}$ (i.e. $\vec{v}_b \simeq \vec{v}_\gamma = \vec{v}_{\gamma b}$). After summing up Eqs. (2.16) and (2.17) to eliminate the scattering terms we arrive at the following equation

$$\partial_\tau \vec{v}_{\gamma b} + \frac{\mathcal{H}R_b}{R_b + 1} \vec{v}_{\gamma b} = \frac{R_b}{R_b + 1} \frac{\vec{J} \times \vec{B}}{\rho_b a^4} - \frac{\vec{\nabla} \delta_\gamma}{4(R_b + 1)} - \vec{\nabla} \phi + \nu_{th} \nabla^2 \vec{v}_{\gamma b}, \quad (2.22)$$

where ν_{th} is the coefficient of thermal diffusivity of the baryon-photon system:

$$\nu_{th} = \frac{4}{15(R_b + 1)} \lambda_{\gamma e}, \quad \lambda_{\gamma e} = \frac{a_0}{\tilde{n}_e \sigma_{\gamma e} a}, \quad \sigma_{\gamma e} = \frac{8}{3} \pi \left(\frac{e^2}{m_e^2} \right), \quad (2.23)$$

and $R_b(\tau)$ is related to the sound speed of the plasma and it is defined as

$$R_b(\tau) = \frac{3\rho_b}{4\rho_\gamma} = 0.629 \left(\frac{\omega_{b0}}{0.02258} \right) \left(\frac{z+1}{1091} \right)^{-1}, \quad c_{sb}(\tau) = \frac{1}{\sqrt{3[R_b(\tau) + 1]}}. \quad (2.24)$$

In Eq. (2.18) ρ_b and δ_b denote, respectively, the mass density of baryons and its inhomogeneities:

$$\rho_b = m_e n_e + m_i n_i, \quad \delta_b = \frac{\delta \rho_b}{\rho_b}. \quad (2.25)$$

In the Λ CDM case the vector modes of the geometry are absent. Strictly speaking the flow considered here is not only compressible (i.e. $\vec{\nabla} \cdot \vec{v}_{\gamma b} \neq 0$) but also irrotational (i.e. $\vec{\nabla} \times \vec{v}_{\gamma b} = 0$). Since $\vec{v}_{\gamma b}$ is irrotational it can be written as $\vec{v}_{\gamma b} = \vec{\nabla} u_{\gamma b}$. As an example the equations for $u_{\gamma b}$ and the magnetic diffusivity equations can be written as:

$$\partial_\tau u_{\gamma b} + \frac{\mathcal{H}R_b}{R_b + 1} u_{\gamma b} = \frac{4\sigma_B - \Omega_B}{R_b + 1} - \frac{\delta_\gamma}{4(R_b + 1)} - \phi + \nu_{th} \nabla^2 u_{\gamma b}, \quad (2.26)$$

$$\partial_\tau \vec{B} = \vec{\nabla} \times [\vec{\nabla} u_{\gamma b} \times \vec{B}] + \lambda \nabla^2 \vec{B}, \quad (2.27)$$

⁵In the present analysis we shall assume a vanishing external gradient in the concentration of the charged species even if this requirement could be relaxed leading, presumably, to an effective evolution of the thermal and magnetic diffusivity coefficients with the physical scale (see, e.g. [26, 27]).

where, according to the notations and conventions of appendix A we used that:

$$\vec{J} \times \vec{B} = \frac{4}{3}a^4\rho_\gamma\left[\vec{\nabla}\sigma_B - \frac{1}{4}\vec{\nabla}\Omega_B\right]. \quad (2.28)$$

Equations (2.26) and (2.27) show that the incompressible closure is not consistent with the presence of scalar inhomogeneities; the adiabatic closure seems to be more appropriate and it is at least consistent with the hydromagnetic equations and with the Λ CDM paradigm. The adiabatic closure amounts to requiring that the fluctuation of the specific entropy (i.e. the ratio between the entropy density of the photons and the concentrations of the baryons) vanishes; this condition is equivalent to the requirement that $\delta_\gamma = 4\delta_b/3$.

As in the case of the one-fluid hydromagnetic equations in flat space⁶, different closures could be adopted (such as the isothermal closure or the closure with constant Ohmic current). While it could be interesting to explore other closures, the one suggested here is definitively better motivated from the viewpoint of the large-scale initial conditions imposed, at early times, on the Einstein-Boltzmann hierarchy [15].

In the dynamo theory the compressible closure is usually adopted (see, e.g. [1, 2, 3]) and the same is true when the kinetic Reynolds number is very large (see, e.g. [32]). However, after electron-positron annihilation $R_{\text{kin}} < 1$ while $Pr_{\text{magn}} > R_{\text{magn}} \gg 1$ [33]. The hypothesis of primeval turbulence (implying the largeness of the kinetic Reynolds number) has been a recurrent theme since the first speculations on the origin of the light nuclear elements. The implications of turbulence for galaxy formation have been pointed out in the fifties by Von Weizsäcker and Gamow [34]. They have been scrutinized in the sixties and early seventies by various authors [35] (see also [36, 37] and discussions therein). In the eighties it has been argued [38] that first-order phase transitions in the early Universe, if present, can provide a source of kinetic turbulence and, hopefully, the possibility of inverse cascades which could lead to an enhancement of the correlation scale of a putative large-scale magnetic field, as discussed in [31, 39, 40]. The extension of the viewpoint conveyed in the present analysis to earlier times (and larger temperatures) is not implausible but shall not be attempted here.

3 Predecoupling flow

The solution of the evolution equations (2.16)–(2.19) determines the correlation functions of the velocity field and the predecoupling flow. By taking the derivative of Eq. (2.19) with respect to the conformal time coordinate τ and by using Eq. (2.22) a well known second-order equation for $\delta_\gamma(k, \tau)$ can be obtained and solved with different methods. In particular, within the WKB approximation [41, 42, 43, 44], the solution for $\delta_\gamma(k, \tau)$ and $u_{\gamma b}(k, \tau)$ reads,

⁶In the context of flat-space magnetohydrodynamics the adiabatic closure amounts to a slightly different requirement, i.e. $\partial_\tau[p\rho_m^{-\kappa}] = 0$; ρ_m is what we called ρ_b (i.e. the mass density of the fluid) and κ is the adiabatic index (i.e. the ratio of the heat capacity at constant pressure and constant volume) [29].

in Fourier space:

$$\delta_\gamma(k, \tau) = -\frac{4}{3c_{\text{sb}}^2}\psi_{\text{m}}(\vec{k}) + \sqrt{c_{\text{sb}}}\mathcal{M}_{\mathcal{R}}(k, \tau) \cos[kr_{\text{s}}(\tau)]e^{-k^2/k_{\text{d}}^2} = \frac{4}{3}\delta_{\text{b}}(k, \tau), \quad (3.1)$$

$$u_{\gamma\text{b}}(k, \tau) = -\frac{1}{k}\overline{\mathcal{M}}_{\mathcal{R}}(\tau)\mathcal{R}_*(\vec{k}) \sin[kr_{\text{s}}(\tau)]e^{-k^2/k_{\text{d}}^2}, \quad (3.2)$$

where $r_{\text{s}}(\tau)$ is the sound horizon and $k_{\text{d}}(\tau)$ defines the time-dependent scale of diffusive damping, i.e.

$$r_{\text{s}}(\tau) = \int_0^\tau c_{\text{sb}}(\tau') d\tau' = \int_0^\tau \frac{d\tau'}{\sqrt{3[R_{\text{b}}(\tau') + 1]}}, \quad \frac{1}{k_{\text{d}}^2(\tau)} = \frac{2}{5} \int_0^\tau \lambda_{\gamma\text{e}}(\tau') c_{\text{sb}}^2(\tau') d\tau'. \quad (3.3)$$

The functions $\psi_{\text{m}}(\vec{k}, \tau) = T_{\mathcal{R}}(\tau)\mathcal{R}_*(k)$, $\mathcal{M}_{\mathcal{R}}(k, \tau)$ and $\overline{\mathcal{M}}_{\mathcal{R}}(\tau)$

$$\mathcal{M}_{\mathcal{R}}(k, \tau) = \frac{4}{3^{3/4}}\left(\frac{1}{c_{\text{sb}}^2} - 2\right)T_{\mathcal{R}}(\tau)\mathcal{R}_*(k), \quad \overline{\mathcal{M}}_{\mathcal{R}}(\tau) = \frac{1 + 3R_{\text{b}}}{\sqrt{3}(1 + R_{\text{b}})^{3/4}}T_{\mathcal{R}}(\tau), \quad (3.4)$$

are all defined in terms of $T_{\mathcal{R}}(\tau)$ which is a simplified form of the transfer function of curvature perturbations discussed, for instance, in appendix B of Ref. [45]:

$$T_{\mathcal{R}}(\tau) = 1 - \frac{\mathcal{H}}{a^2} \int_0^\tau a^2(\tau') d\tau' = 1 - \frac{H}{a} \int_0^\tau a(t') dt'. \quad (3.5)$$

Note that, for $\tau \rightarrow \tau_*$ (where τ_* denotes the last scattering) $T_{\mathcal{R}}(\tau_*) \simeq -(3/5)$. If $k\tau \ll 1$ we have, from Eq. (3.1), that $\delta_\gamma \rightarrow -8\psi_{\text{m}}/3$, $c_{\text{sb}} \rightarrow 1/\sqrt{3}$ and $\overline{\mathcal{M}}_{\mathcal{R}}(\tau_*) \rightarrow \sqrt{3}/5$. Since the curvature perturbations are distributed as in Eqs. (2.2)–(2.3), the correlation function of the velocity for unequal times can be written as:

$$\langle v_i(\vec{q}, \tau) v_j(\vec{p}, \tau') \rangle = \frac{q_i q_j}{q^2} \mathcal{U}(q, |\tau - \tau'|) \delta^{(3)}(\vec{q} + \vec{p}), \quad \mathcal{U}(q, |\tau - \tau'|) = v(q) \delta(\tau - \tau'), \quad (3.6)$$

where, to avoid confusions with vector indices, the subscript “ γb ” has been suppressed. The function $v(q)$ appearing in Eq. (3.6) is

$$v(q) = \tau_{\text{c}} \mathcal{V}(q), \quad \mathcal{V}(q) = \overline{\mathcal{M}}_{\mathcal{R}}^2(\tau_*) \frac{2\pi^2}{q^3} \mathcal{P}_{\mathcal{R}}(q) \sin^2[qr_{\text{s}}(\tau_*)] e^{-2q^2/q_{\text{d}}^2}. \quad (3.7)$$

The correlation time τ_{c} is, by definition, the smallest time-scale when compared with other characteristic times arising in the problem. Because of the exponential suppression of the velocity correlation function for $\tau > \tau_{\text{d}}$ (where τ_{d} denotes the Silk time [30]), τ_{c} approximately coincides with τ_{d} . The form of the correlator given in Eq. (3.6) is characteristic of Markovian conducting fluids [17, 21]. As a consequence of the smallness of τ_{c} the velocity correlator for unequal times can be approximated with its Markovian form given in Eq. (3.7). The velocity field is exponentially suppressed for $\tau > \tau_{\text{c}} = 1/(k_{\text{max}}^2 \nu_{\text{th}})$ where $\tau_{\text{c}} \simeq 1.95 \times 10^{-6} (z_* + 1)^2 \text{ Mpc}$, $z_* \simeq 1090$ and, typically, $k_{\text{max}} \simeq k_{\text{d}}$. Note that $v(q)$ contains τ_{c} and it does not have the

same dimensions of $\mathcal{V}(q)$. Defining, for immediate convenience, $\mathcal{P}_v(q) = q^3 v(q)/(2\pi^2)$, the velocity correlator can be expressed in physical space as

$$\langle v_i(\vec{x}, \tau) v_j(\vec{y}, \tau') \rangle = \left\{ V_T(r) \delta_{ij} + \left[V_L(r) - V_T(r) \right] \frac{r_i r_j}{r^2} \right\} \delta(\tau - \tau'), \quad (3.8)$$

$$\begin{aligned} V_T(r) &= \int \frac{dq}{q} \mathcal{P}_v(q) \left[\frac{1}{q^3 r^3} \sin qr - \frac{1}{q^2 r^2} \cos qr \right], \\ V_L(r) &= \int \frac{dq}{q} \mathcal{P}_v(q) \left[-\frac{2}{q^3 r^3} \sin qr + \frac{2}{q^2 r^2} \cos qr + \frac{\sin qr}{qr} \right], \end{aligned} \quad (3.9)$$

where $r = |\vec{x} - \vec{y}|$. Thanks to the explicit form of the fluid flow, Eqs. (2.21) and (2.27) can be solved by neglecting all the terms which are of higher order in the magnetic field intensity and by focusing the attention on the coupling of the compressible flow to the magnetic field; by following Ref. [17, 21] Eqs. (2.21)–(2.27) can be solved iteratively as

$$B_i(\vec{k}, \tau) = \sum_{n=0}^{\infty} B_i^{(n)}(\vec{k}, \tau), \quad \mathcal{G}_k(y) = e^{-k^2 \lambda y}, \quad (3.10)$$

$$\begin{aligned} B_i^{(n+1)}(\vec{k}, \tau) &= \frac{(-i)}{(2\pi)^{3/2}} \int_0^\tau \mathcal{G}_k(\tau - \tau_1) d\tau_1 \int d^3 q \int d^3 p \delta^{(3)}(\vec{k} - \vec{q} - \vec{p}) \\ &\times \epsilon_{m n i} \epsilon_{a b n} (q_m + p_m) v_a(\vec{q}, \tau_1) B_b^{(n)}(\vec{p}, \tau_1), \end{aligned} \quad (3.11)$$

where, for simplicity, λ is assumed to be constant in time; moreover, as in Eq. (2.27), $v_a(\vec{x}, \tau) = \partial_a u(\vec{x}, \tau)$. The magnetic field can then be averaged over the flow by using either Eq. (3.6) or Eq. (3.8) (depending if we work in real space or in Fourier space). From Eq. (3.11) the first few terms of the recursion are $B_i^{(0)}(\vec{k}, \tau)$, $B_i^{(1)}(\vec{k}, \tau)$ and $B_i^{(2)}(\vec{k}, \tau)$. The first term is $B_i^{(0)}(\vec{k}, \tau) = \mathcal{G}_k(\tau) B_i(\vec{k})$ where $B_i(\vec{k})$ parametrizes the initial stochastic magnetic field. Denoting with $H_i(\vec{k}, \tau)$ the magnetic field averaged over the fluid flow, the terms containing an odd number of velocities will be zero while the correlators containing an even number of velocities do not vanish i.e. $\langle B_i^{(2n+1)} \rangle = H_i^{(2n+1)} = 0$ and $\langle B_i^{(2n+2)} \rangle = H_i^{(2n+2)} \neq 0$. So, for instance, $\langle B_i^{(1)} \rangle = 0$ while $\langle B_i^{(2)} \rangle = H_i^{(2)}$ is

$$\begin{aligned} H_i^{(2)}(\vec{k}, \tau) &= \frac{(-i)^2}{(2\pi)^3} \int d^3 q \int d^3 p \int d^3 q' \int d^3 p' \delta^{(3)}(\vec{k} - \vec{q} - \vec{p}) \delta^{(3)}(\vec{p} - \vec{q}' - \vec{p}') \\ &\times \int_0^\tau d\tau_1 \mathcal{G}_k(\tau - \tau_1) \int_0^{\tau_1} d\tau_2 \mathcal{G}_p(\tau_1 - \tau_2) (q_m + p_m) (q'_m + p'_m) \epsilon_{b m' n'} \epsilon_{a' b' n'} \epsilon_{m n i} \epsilon_{a b n} \\ &\times \langle v_{a'}(\vec{q}', \tau_2) v_a(\vec{q}, \tau_1) \rangle B_{b'}(\vec{p}'). \end{aligned} \quad (3.12)$$

After averaging the whole series of Eq. (3.10) term by term the obtained result can be resummed and written as:

$$H_i(\vec{k}, \tau) = \langle B_i^{(0)}(\vec{k}, \tau) \rangle + \langle B_i^{(2)}(\vec{k}, \tau) \rangle + \langle B_i^{(4)}(\vec{k}, \tau) \rangle + \dots = e^{-k^2 \bar{\lambda} \tau} B_i(\vec{k}). \quad (3.13)$$

where the magnetic diffusivity coefficient $\lambda = 1/(4\pi\sigma)$ inherits a modification stemming from the large-scale flow:

$$\bar{\lambda} = \lambda + v_0, \quad v_0 = \frac{\tau_c}{3} \overline{\mathcal{M}}_{\mathcal{R}}^2(\tau_*) \int \frac{dk}{k} \mathcal{P}_{\mathcal{R}}(k) \sin^2(k/k_*) e^{-2k^2/k_*^2}, \quad (3.14)$$

and $k_* = 1/r_s(\tau_*)$. The averaging suggested here has been explored long ago in the related context of acoustic turbulence by Vainshtein [21]. Prior to decoupling, however, both $\langle v^2 \rangle \propto \mathcal{A}_{\mathcal{R}} \ll 1$ and $R_{\text{kin}} \ll 1$. The iterative solution indicated in Eq. (3.13) seems then to be better defined, in the present case, than in the a kinetically turbulent plasma with strong inhomogeneities. The presence of v_0 defines a new diffusion scale associated with the large-scale flow and this scales can be estimated from Eq. (3.14) as

$$k_M d_A \simeq \sqrt{\frac{d_A}{\tau_c}} \sqrt{\frac{6(1-n_s)}{\mathcal{A}_{\mathcal{R}} \mathcal{M}_{\mathcal{R}}^2(\tau_*)}} (d_A k_p)^{(n-1)/2}, \quad (3.15)$$

where subscript M stands for “Markovian” and where d_A denotes the (comoving) angular diameter distance to last scattering: since k_M depends on time, it has been evaluated at last scattering. In the appendix B the analysis in the non-Markovian approach will be outlined, under some simplifying approximations, with particular attention to the determination of the diffusive scale.

4 Evolution of the power spectra

Instead of averaging the magnetic field intensities, as explored in the previous section, it is more practical to study directly the evolution equations of the magnetic power spectra with the aim of connecting them to the two-point function of the velocity field. In Fourier space the magnetic power spectrum can be defined as

$$\langle B_i(\vec{k}, \tau) B_j(\vec{k}', \tau) \rangle = M_{ij}(\vec{k}, \tau) \delta^{(3)}(\vec{k} + \vec{k}'), \quad M_{ij}(\vec{k}, \tau) = P_{ij}(\hat{k}) M(\vec{k}, \tau), \quad (4.1)$$

where $P_{ij}(\hat{k}) = (\delta_{ij} - \hat{k}_i \hat{k}_j)$. The corresponding power spectrum in physical space is given by

$$M_{ij}(r, \tau) = M_T(r, \tau) \delta_{ik} + [M_L(r, \tau) - M_T(r, \tau)] \frac{r_i r_k}{r^2}, \quad (4.2)$$

$$\frac{\partial M_L}{\partial r} + \frac{2}{r} (M_L - M_T) = 0, \quad (4.3)$$

where Eq. (4.3) is a consequence of the transversality condition, i.e. $\partial_i M^{ij} = 0$. The evolution equation of the magnetic power spectrum in the compressible limit stems from Eq. (2.27) and can be obtained by taking the conformal time derivative of $M_{ij}(k, \tau)$ as defined in Eq. (4.1). The result of this step is the sum of two terms: each of the terms contains the product of the time derivative of the magnetic field with the magnetic field itself for different wavenumbers. The time derivative of the magnetic field can be eliminated by means of Eq. (2.27) while the magnetic field itself can be written in terms of the integral equation derived always from Eq. (2.27). The result of this procedure is, in the compressible case,

$$\begin{aligned} \partial_\tau M_{ij} + 2\lambda k^2 M_{ij} = & (-i)^2 \int \frac{d^3 q}{(2\pi)^{3/2}} \int \frac{d^3 q'}{(2\pi)^{3/2}} k_m k_{m'} \epsilon_{imn} \epsilon_{abn} \epsilon_{j m' n'} \epsilon_{a' b' n'} \times \\ & \times \langle v_a(\vec{q}, \tau) v_{a'}(\vec{q}', \tau') \rangle M_{bb'}(\vec{p}, \vec{p}', \tau, \tau'), \end{aligned} \quad (4.4)$$

where $\vec{p} = (\vec{k} + \vec{q})$ and $\vec{p}' = (\vec{k} + \vec{q}')$. Using now the explicit form of the velocity correlator given in Eq. (3.6) and taking the trace of both sides with respect to the free tensor indices i and j the following integrodifferential equation is readily obtained

$$\partial_\tau M + 2k^2 \bar{\lambda} M = \int \frac{d^3 q}{(2\pi)^3} v(q) M(p, \tau) \left[k^2 - \frac{(\vec{k} \cdot \vec{p})^2}{p^2} + 2 \frac{(\vec{k} \cdot \vec{q})(\vec{k} \cdot \vec{p})(\vec{q} \cdot \vec{p})}{q^2 p^2} \right], \quad (4.5)$$

where, for convenience, \vec{p} has been kept explicitly but it is in fact $\vec{p} = (\vec{k} - \vec{q})$. Note that $\bar{\lambda}$ is the diffusivity coefficient accounting also for the effects of the flow (see Eq. (3.14)).

The same derivation can be performed in the incompressible case but bearing in mind that the correlation function of the velocity field will have a different form which is determined by the transversality of the flow. The analog of Eq. (4.5) in the case of the incompressible closure is given by:

$$\partial_\tau M + 2k^2 \bar{\lambda} M = \int \frac{d^3 q}{(2\pi)^3} \tilde{v}(q) M(\vec{p}, \tau) \left[k^2 - \frac{(\vec{k} \cdot \vec{q})(\vec{k} \cdot \vec{p})(\vec{p} \cdot \vec{q})}{q^2 p^2} \right], \quad (4.6)$$

which coincides with the equation derived by Kazantsev [17] except for the presence of $1/(2\pi)^3$ coming from the different conventions on the Fourier transforms (we follow here the conventions spelled out explicitly in appendix A). To correctly interpret Eq. (4.6) it is essential to point out that the velocity field parametrized by $\tilde{v}(q)$ is now solenoidal (rather than irrotational as in the case of Eq. (4.5)). To derive Eq. (4.6) we used the analog of Eq. (3.6) but valid for the standard incompressible closure, i.e.

$$\langle \tilde{v}_i(\vec{q}, \tau) \tilde{v}_j(\vec{p}, \tau') \rangle = \left(\delta_{ij} - \frac{q_i q_j}{q^2} \right) \tilde{\mathcal{U}}(q, |\tau - \tau'|) \delta^{(3)}(\vec{q} + \vec{p}), \quad \tilde{\mathcal{U}}(q, |\tau - \tau'|) = \tilde{v}(q) \delta(\tau - \tau'). \quad (4.7)$$

The results of Eqs. (4.6) and (4.7) are a consistency check of the whole approach. For sake of comparison, the incompressible equations (derived from Eq. (4.6)) will be sometimes reported after the corresponding results valid in the compressible case. It is finally useful to point out that in Eqs. (4.5) and (4.6) the diffusivity coefficient is $\bar{\lambda}$ (and not λ itself). This property has been already derived, at the level of the field intensities, in Eqs. (3.13)–(3.14). Strictly speaking Eqs. (4.5) and (4.6) describe the evolution of the magnetic power spectra averaged over the fluid flow. In this respect following the notations of Kazantsev [17] (see also [21]) it is sometimes useful to distinguish the “double” stochastic average $\langle\langle \dots \rangle\rangle$ (over the velocity and over the magnetic field) from the single average (valid either for the velocity or for the magnetic field at the level of the correlators). To preserve a certain simplicity in the notations the double stochastic average has been avoided.

4.1 Diffusive approximation

The integrodifferential equations (4.5) and (4.6) can be studied, with two complementary approaches, either in Fourier space or in physical space. By taking the limit $q \rightarrow 0$ (while k

is held fixed), Eqs. (4.5)–(4.6) lead to a diffusion equation which is very similar to the one often encountered in the dynamics of wave turbulence [46, 47]. Equations (4.5)–(4.6) can also be transformed into a Schrödinger-like equation in physical space where the analog of the wavefunction is related to the power spectrum introduced in Eq. (4.2). In what follows the relevant evolution equations will be derived in these two complementary approaches. The attention will be focused on the compressible mode, however, as a cross-check, the same derivations have been also performed with the incompressible closure. In the latter case the equations reproduce exactly the results reported in [17]. This represents an important cross-check for the consistency of the whole procedure.

In the diffusive limit Eq. (4.5) is expanded for $\epsilon = |q/k| \ll 1$. The integrand appearing at the right hand side of Eq. (4.5) written in terms of ϵ becomes

$$\begin{aligned} M(\vec{p}, \tau) & \left[k^2 - \frac{(\vec{k} \cdot \vec{p})^2}{p^2} + 2 \frac{(\vec{k} \cdot \vec{q})(\vec{k} \cdot \vec{p})(\vec{q} \cdot \vec{p})}{q^2 p^2} \right] \\ & = k^2 M \left(k \sqrt{1 - 2\epsilon x + \epsilon^2} \right) \left[1 - \frac{(1 - \epsilon x)^2}{1 - 2\epsilon x + \epsilon^2} + 2 \frac{x(1 - \epsilon x)(x - \epsilon)}{1 - 2\epsilon x + \epsilon^2} \right]. \end{aligned} \quad (4.8)$$

where $x = \cos \vartheta$ and $\vartheta = (\hat{q} \cdot \hat{k})$. After expanding Eq. (4.8) in powers of ϵ , direct integration over x between -1 and 1 gives

$$\frac{4}{3} k^2 M + \frac{2k^2}{15} \left[2M + 6k \frac{\partial M}{\partial k} + 3k^2 \frac{\partial^2 M}{\partial k^2} \right] \epsilon^2. \quad (4.9)$$

Using the results of Eqs. (4.8) and (4.9), Eq. (4.5)

$$\frac{\partial M}{\partial \tau} + 2\lambda k^2 M = \gamma \left[2M + 6k \frac{\partial M}{\partial k} + 3k^2 \frac{\partial^2 M}{\partial k^2} \right], \quad v_2 = \frac{1}{3!} \int \frac{d^3 q}{(2\pi)^3} q^2 v(q), \quad (4.10)$$

where $\gamma = 2v_2/5$. From the explicit expression of $v(q)$ it is possible to estimate γ by replacing $\sin^2 \rightarrow 1/2$, $\overline{\mathcal{M}}_{\mathcal{R}}^2(\tau_*) \rightarrow 3/25$, and by fixing the upper limit of integration at k_d :

$$\gamma = \frac{\mathcal{A}_{\mathcal{R}}}{30(n_s + 1)} \overline{\mathcal{M}}^2(\tau_*) \left(\frac{k_d}{k_p} \right)^{n_s + 1} q_p^2 \tau_c, \quad \eta = \gamma \tau, \quad (4.11)$$

where, as already mentioned, $\overline{\mathcal{M}}^2(\tau_*) \rightarrow 3/25$; the introduction of the variable η will be useful in section 5 since it will simplify the expressions of the Laplace transforms. Choosing, for sake of simplicity $k_d \sim 0.1 \text{ Mpc}^{-1}$ as well as the fiducial set of parameters of Eqs. (2.3) and (2.4) η can be explicitly estimated; for $\tau \simeq \mathcal{O}(\tau_*)$ we have that $\eta \simeq 6.04 \times 10^{-11}$.

As anticipated after Eqs. (4.6)–(4.7), the same kind of equation obtained in the compressible limit can be derived when the fluid flow is incompressible. Indeed, from the right hand side of Eq. (4.6) we have

$$M(\vec{p}, \tau) \left[k^2 - \frac{(\vec{k} \cdot \vec{q})(\vec{k} \cdot \vec{p})(\vec{p} \cdot \vec{q})}{q^2 p^2} \right] = k^2 M \left(k \sqrt{1 - 2\epsilon x + \epsilon^2} \right) \left[1 - \frac{x(1 - \epsilon x)(x - \epsilon)}{1 - 2\epsilon x + \epsilon^2} \right]. \quad (4.12)$$

By expanding Eq. (4.12) for small ϵ and by integrating over x between -1 and 1 we have

$$\frac{4}{3}k^2 M + \frac{2k^2}{15} \left[4M + 2k \frac{\partial M}{\partial k} + k^2 \frac{\partial^2 M}{\partial k^2} \right] \epsilon^2, \quad (4.13)$$

implying, from Eq. (4.6),

$$\frac{\partial M}{\partial \tau} + 2\lambda k^2 M = \frac{2}{5} \tilde{v}_2 \left[4M + 2k \frac{\partial M}{\partial k} + k^2 \frac{\partial^2 M}{\partial k^2} \right], \quad \tilde{v}_2 = \frac{1}{3!} \int \frac{d^3 q}{(2\pi)^3} q^2 \tilde{v}(q). \quad (4.14)$$

Apart from the slightly different conventions, Eq. (4.14) reproduces the corresponding results of [17] and corroborates the correctness of the results obtained here in the compressible case.

4.2 Schrödinger-like equations

In physical space Eq. (4.5) can be transformed into a Schrödinger-like equation whose generalized wavefunction is related to the magnetic power spectrum. Since, by definition,

$$M(r, \tau) = \frac{1}{(2\pi)^{3/2}} \int d^3 k e^{-i\vec{k} \cdot \vec{r}} M(k, \tau), \quad (4.15)$$

both sides of Eq. (4.5) can be multiplied by $\exp[-i\vec{k} \cdot \vec{r}]$ and then integrated over $d^3 k$. Recalling that $\vec{k} = \vec{q} + \vec{p}$, after simple algebra the evolution of $M(r, \tau)$ can be obtained in physical space:

$$\partial_\tau M - 2\bar{\lambda} \frac{\partial^2 M}{\partial r_a \partial r^a} = -\frac{\partial^2 (v M)}{\partial r_a \partial r^a} + \frac{\partial^2}{\partial r_i \partial r_j} [G_{ij} v] - 2 \frac{\partial^2}{\partial r_i \partial r_j} [G_{ik} F_{kj}], \quad (4.16)$$

where it is understood that $M = M(r, \tau)$. According to Eq. (3.8), v coincides with the trace of the correlation function of the velocity in real physical space, i.e. $v = (2V_T + V_L)$. The tensors G_{ij} and F_{ij} are:

$$G_{ij} = \bar{M} \delta_{ij} + (M - 3\bar{M}) \frac{r_i r_j}{r^2}, \quad F_{ij} = \frac{v - V_L}{2} \delta_{ij} + (3V_L - v) \frac{r_i r_j}{2r^2}, \quad (4.17)$$

where $M(r, \tau)$, $\bar{M}(r, \tau)$ and $M_L(r, \tau)$ are related as follows:

$$\bar{M}(r, \tau) = \frac{M_L(r, \tau)}{2} = \frac{1}{r^3} \int_0^r x^2 M(x, \tau) dx. \quad (4.18)$$

Since for a generic function $f(r, \tau)$ it is easy to show that

$$\frac{\partial^2 f}{\partial r_a \partial r^a} = \frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 \frac{\partial f}{\partial r} \right], \quad \frac{\partial^2}{\partial r_i \partial r_j} [r_i r_j f] = \frac{1}{r^2} \frac{\partial^2}{\partial r^2} [r^4 f]. \quad (4.19)$$

Eq. (4.16) can be transformed as:

$$\frac{\partial \bar{M}}{\partial \tau} + 2(V_L - \bar{\lambda}) \frac{\partial^2 \bar{M}}{\partial r^2} + 2 \left[\frac{\partial V_L}{\partial r} + \frac{4}{r} (V_L - \bar{\lambda}) \right] \frac{\partial \bar{M}}{\partial r} + 2 \left[\frac{1}{r} \frac{\partial}{\partial r} (v + V_L) + \frac{1}{r^2} (3V_L - v) \right] \bar{M} = 0, \quad (4.20)$$

which can also be written in the following equivalent form:

$$\frac{\partial \overline{M}}{\partial \tau} + \frac{2}{r^4} \frac{\partial}{\partial r} \left[r^4 (V_L - \overline{\lambda}) \frac{\partial \overline{M}}{\partial r} \right] + \frac{2}{r} \left[\frac{\partial}{\partial r} (v + V_L) + \frac{1}{r} (3V_L - v) \right] \overline{M} = 0. \quad (4.21)$$

By performing the same derivation in the case of Eq. (4.6), the analog of Eq. (4.21) becomes

$$\frac{\partial \overline{M}}{\partial \tau} + 2(\overline{V} - \overline{\lambda}) \frac{\partial^2 \overline{M}}{\partial r^2} + 2(v - 4\overline{\lambda} + \overline{V}) \frac{1}{r} \frac{\partial \overline{M}}{\partial r} + \frac{2}{r} \overline{M} \frac{\partial v}{\partial r} = 0, \quad (4.22)$$

where, in this case⁷

$$\overline{V} = \frac{1}{r^3} \int_0^r x^2 v(x) dx = \frac{V_L}{2}, \quad 2V_T + V_L = v. \quad (4.23)$$

Equation (4.23) coincides with the analog equation obtained in [17]. It is finally appropriate to mention that Eq. (4.21) coincides with the equation derived by Vainshtein and Kichatinov in [48] once the relevant functions are appropriately renamed. The correspondence is such that $V_T \rightarrow T_{NN}$, $V_L \rightarrow T_{LL}$, and $B_{LL} \rightarrow M_L = 2\overline{M}$. Equations (4.21) and (4.6) can be put in different but equivalent forms like, for instance,

$$\frac{\partial \overline{M}}{\partial \tau} + 2\hat{\mathcal{L}}_f(r) \overline{M}(r, \tau) = 0, \quad (4.24)$$

$$\frac{\partial \overline{M}}{\partial \tau} + 2\hat{\mathcal{L}}_g(r) \overline{M}(r, \tau) = 0, \quad (4.25)$$

where Eq. (4.24) holds for an incompressible flow while Eq. (4.25) holds for a compressible flow. In the incompressible case the linear operator $\hat{\mathcal{L}}_f(r)$ takes the form

$$\hat{\mathcal{L}}_f(r) = -f(r) \frac{\partial^2}{\partial r^2} - f_1(r) \frac{\partial}{\partial r} + f_2(r), \quad (4.26)$$

where the three functions $f(r)$, $f_1(r)$ and $f_2(r)$ are defined as

$$f(r) = \overline{\lambda} - \overline{V}, \quad f_1(r) = \frac{1}{r} [4\overline{\lambda} - \overline{V} - v], \quad f_2(r) = \frac{1}{r} \frac{\partial v}{\partial r}. \quad (4.27)$$

Conversely, in the compressible case, the linear operator $\hat{\mathcal{L}}_g(r)$ is

$$\hat{\mathcal{L}}_g(r) = -g(r) \frac{\partial^2}{\partial r^2} - g_1(r) \frac{\partial}{\partial r} + g_2(r), \quad (4.28)$$

and the explicit form of the functions $g(r)$, $g_1(r)$ and $g_2(r)$ is

$$\begin{aligned} g(r) &= \overline{\lambda} - V_L(r), & g_1(r) &= \frac{\partial}{\partial r} (\overline{\lambda} - V_L) + \frac{4}{r} (\overline{\lambda} - V_L), \\ g_2(r) &= \frac{1}{r} \frac{\partial}{\partial r} (v + V_L) + \frac{1}{r^2} (3V_L - v) = \frac{2}{r} \frac{\partial}{\partial r} (V_T + V_L) + \frac{2}{r^2} (V_L - V_T). \end{aligned} \quad (4.29)$$

⁷We recall that, in the incompressible case, the explicit forms of the velocity correlator in real space is the same as the one given in Eq. (3.8) but with the difference that the transversality condition applied to the velocity field implies that $\partial_r V_L + 2(V_L - V_T)/r = 0$; the latter condition leads to the first expression of Eq. (4.23).

By focusing the attention on the compressible case of Eqs. (4.25) and (4.28)–(4.29), Eq. (4.25) can be rewritten by eliminating the first derivative with respect to r :

$$\frac{\partial \Psi}{\partial \tau} = 2g \frac{\partial^2 \Psi}{\partial r^2} - 2W(r)\Psi, \quad \overline{M}(r, \tau) = Q(r)\Psi(r, \tau); \quad (4.30)$$

the potential $W(r)$ and $Q(r)$ are

$$W(r) = g \left[\frac{1}{2} \left(\frac{g_1}{g} \right)' + \frac{1}{4} \left(\frac{g_1}{g} \right)^2 \right] + g_2(r), \quad \frac{Q'}{Q} = -\frac{g_1}{2g}, \quad (4.31)$$

where the prime denotes, in Eqs. (4.31) and (4.32) a derivation with respect to r . Defining, for immediate convenience, $g_1/g = F'/F$, Eq. (4.31) implies, together with the definition of $g_1(r)$ of Eq. (4.29),

$$W(r) = g(r) \frac{(\sqrt{F})''}{\sqrt{F}} + g_2(r), \quad Q(r) = \frac{1}{\sqrt{F(r)}}, \quad F(r) = r^2 \sqrt{g(r)}. \quad (4.32)$$

5 Solutions for the compressible mode

5.1 Kinetic equations for the power spectra

It is interesting to solve the equations obtained in the previous section by assuming that the magnetic power spectra are assigned at a specific pivot scale conventionally denoted by k_L [14]

$$\langle B_i(\vec{q}, \tau) B_j(\vec{p}, \tau) \rangle = \frac{2\pi^2}{q^3} \mathcal{P}_B(k, \tau) P_{ij}(\hat{q}), \quad P_{ij}(\hat{q}) = \delta_{ij} - \hat{q}_i \hat{q}_j, \quad (5.1)$$

where $\hat{q}_i = q_i/q$. The evolution equation of $\mathcal{P}_B(k, \tau)$ will be solved by imposing the following boundary conditions

$$\mathcal{P}(k, 0) = A_B \left(\frac{k}{k_L} \right)^{n_B - 1}, \quad \mathcal{P}(k_L, \tau) = A_B G(\tau). \quad (5.2)$$

Even if $G(\tau)$ will be left unspecified a simple choice would imply $G(\tau) \propto \delta(\tau)$. The evolution equation for $\mathcal{P}_B(k, \tau)$ stems from Eq. (4.9) by comparing the parametrization of Eq. (4.1) with the one of Eq. (5.1):

$$\frac{\partial \mathcal{P}_B}{\partial \tau} + 2k^2 \lambda P_B = \gamma \left[20 \mathcal{P}_B - 12 k \frac{\partial \mathcal{P}_B}{\partial k} + 3 k^2 \frac{\partial^2 \mathcal{P}_B}{\partial k^2} \right]. \quad (5.3)$$

Depending on the value of the conductivity there are two regimes which can be identified. In the first regime the magnetic diffusivity is subleading (i.e. $k^2 \lambda \ll 10\gamma$) and Eq. (5.3) becomes:

$$\frac{\partial \mathcal{P}_B}{\partial \tau} = \gamma \left[20 \mathcal{P}_B - 12 k \frac{\partial \mathcal{P}_B}{\partial k} + 3 k^2 \frac{\partial^2 \mathcal{P}_B}{\partial k^2} \right]. \quad (5.4)$$

When the magnetic diffusivity is subleading Eq. (5.4) can be solved with the boundary conditions (5.2) by standard Laplace transform methods. In particular, defining

$$\overline{\mathcal{P}}_{\text{B}}(k, s) = \int_0^\infty e^{-s\tau} \mathcal{P}_{\text{B}}(k, \tau) d\tau = \mathcal{L}[\mathcal{P}_{\text{B}}(k, \tau)], \quad (5.5)$$

Eq. (5.4) together with the first of the two conditions of Eq. (5.2) implies the following (ordinary) differential equation:

$$\frac{d^2 \overline{\mathcal{P}}_{\text{B}}}{dk^2} - \frac{4}{k} \frac{d \overline{\mathcal{P}}_{\text{B}}}{dk} + \left[\frac{20}{3k^2} - \frac{s}{3\gamma k^2} \right] \overline{\mathcal{P}}_{\text{B}} = -\frac{A_{\text{B}}}{3\gamma k^2} \left(\frac{k}{k_{\text{L}}} \right)^{n_{\text{B}}-1}. \quad (5.6)$$

After a standard change of variables Eq. (5.6) can be transformed as

$$\frac{d^2 \overline{\mathcal{Q}}_{\text{B}}}{dz^2} - q^2(s, \gamma) \overline{\mathcal{Q}}_{\text{B}} = -\frac{A_{\text{B}}}{3\gamma} e^{pz} \quad q(s, \gamma) = \frac{1}{\sqrt{3}\gamma} \sqrt{s - \frac{5}{4}\gamma}, \quad p = n_{\text{B}} - \frac{7}{2}. \quad (5.7)$$

where the variables z and $\overline{\mathcal{Q}}_{\text{B}}(z, s)$ are simply:

$$z = \ln(k/k_{\text{L}}), \quad \overline{\mathcal{Q}}_{\text{B}}(z, s) = e^{-5z/2} \overline{\mathcal{P}}_{\text{B}}(z, s). \quad (5.8)$$

The general solution of Eq. (5.7) is

$$\begin{aligned} \overline{\mathcal{Q}}_{\text{B}}(z, s) &= \left[c_1(s) + \frac{A_{\text{B}}}{6\gamma q(p - q(s, \gamma))} \right] e^{qz} + \left[c_2(s) - \frac{A_{\text{B}}}{6\gamma q(s, \gamma)(p + q(s, \gamma))} \right] e^{-q(s, \gamma)z} \\ &- \frac{A_{\text{B}}}{3\gamma} \frac{e^{pz}}{p^2 - q^2(s, \gamma)}. \end{aligned} \quad (5.9)$$

If $z \geq 0$ the second condition of Eq. (5.2) implies that

$$\begin{aligned} c_2(s) &= A_{\text{B}} \left[g(s) + \frac{1}{6\gamma q(s, \gamma)(p + q(s, \gamma))} + \frac{1}{3\gamma(p^2 - q^2(s, \gamma))} \right], \\ c_1(s) &= -\frac{A_{\text{B}}}{6\gamma q(s, \gamma)(p - q(s, \gamma))}, \end{aligned} \quad (5.10)$$

where $g(s)$ simply denotes the Laplace transform of the function $G(\tau)$ which appears in the second relation of Eq. (5.2). Equation (5.10) applies for $z \geq 0$, i.e. $k \geq k_{\text{L}}$. Similarly, if $z \leq 0$, the second condition of Eq. (5.2) together with the requirement of the existence of the inverse Laplace transform implies

$$\begin{aligned} c_1(s) &= A_{\text{B}} \left[g(s) - \frac{1}{6\gamma q(s, \gamma)(p - q(s, \gamma))} + \frac{1}{3\gamma(p^2 - q^2(s, \gamma))} \right], \\ c_2(s) &= \frac{A_{\text{B}}}{6\gamma q(s, \gamma)(p + q(s, \gamma))}. \end{aligned} \quad (5.11)$$

Equation (5.11) applies for $z \leq 0$, i.e. for $k \leq k_{\text{L}}$. The Laplace antitransform can be easily deduced by appropriately choosing the integration contour in the complex plane. In the case $z \leq 0$ we have that $\overline{\mathcal{Q}}_{\text{B}}(z, s)$ is given by:

$$\overline{\mathcal{Q}}_{\text{B}}(z, s) = \frac{A_{\text{B}}}{3\gamma} \left[\frac{e^{q(s, \gamma)z}}{p^2 - q^2(s, \gamma)} + 3\gamma g(s) e^{q(s, \gamma)z} - \frac{e^{pz}}{p^2 - q^2(s, \gamma)} \right]. \quad (5.12)$$

An analog expression can be deduced in the case $z \geq 0$ by choosing the constants of Eq. (5.9) as in Eq. (5.10). Since the explicit expressions of $q(s, \gamma)$ and p have been given in Eq. (5.7) it is easy to see that the first term in Eq. (5.12) has a branch point from the argument of the exponential and a simple pole from the denominator; the second term has only a branch point while the third term has just a simple pole. By taking the inverse Laplace transform of Eq. (5.12) the spectrum can be explicitly obtained. The same procedure described so far can be repeated in the case $z \geq 0$. The general form of $\mathcal{P}_B(z, \tau)$ is then given by:

$$\begin{aligned} \mathcal{P}_B(z, \tau) = & A_B \left[e^{(n_B-1)z} e^{5\gamma\tau/4+3\gamma\tau(n_B-7/2)^2} - e^{5\gamma\tau/4+5z/2} \mathcal{F}(\tau, |z|, n_B) \right. \\ & \left. + |z| e^{5z/2} \int_0^\tau \frac{G(\tau-u)}{\sqrt{12} \pi \gamma u} e^{5\gamma u/4-|z|^2/(12\gamma u)} \frac{du}{u} \right], \end{aligned} \quad (5.13)$$

$$\mathcal{F}(\tau, |z|, n_B) = \int_0^\infty e^{-x\tau} \frac{\sin\left(|z| \sqrt{\frac{x}{3\gamma}}\right)}{x + 3\gamma(n_B - 7/2)^2} dx. \quad (5.14)$$

The first and second terms of Eq. (5.13) come from the contributions containing, respectively, a simple pole and a branch point in the Laplace transform $\overline{Q}_B(z, s)$; the third term in Eq. (5.13) corresponds to the piece containing a simple pole and a branch point. The properties of the solution (5.13) are determined by the value of $\gamma\tau$ already examined in Eq. (4.11). It is useful to note that when $\gamma\tau$ is kept fixed in such a way that $\gamma \rightarrow 0$ and $\tau \rightarrow \infty$ Eq. (5.14) has a definite limit:

$$\lim_{\gamma \rightarrow 0, \tau \rightarrow \infty} \mathcal{F}(\tau, |z|, n_B) = \text{Erf}\left(\frac{|z|}{\sqrt{12\gamma\tau}}\right). \quad (5.15)$$

As already stressed in connection with Eqs. (5.3)–(5.4), the solution (5.13) holds in the limit $k^2\lambda < 10\gamma$. In the opposite limit (i.e. $k^2\lambda > 10\gamma$) it is possible to look for the solution by bootstrapping Eq. (5.13):

$$\mathcal{P}_B(k, \tau) = e^{5\gamma\tau/4+3(n_B-7/2)^2\gamma\tau} \mathcal{G}(k). \quad (5.16)$$

Inserting Eq. (5.16) into Eq. (5.4) and eliminating the first derivatives with respect to k the following equation can be obtained in terms of the rescaled function $\overline{\mathcal{G}}_k = \mathcal{G}(k)/k^2$:

$$\frac{d^2\overline{\mathcal{G}}}{dk^2} + \left[-\frac{2\lambda}{3\gamma} - \frac{\nu^2 - 1/4}{k^2} \right] \overline{\mathcal{G}} = 0, \quad \nu = |n_B - 7/2|. \quad (5.17)$$

The general solution of Eq. (5.17) is given in terms of a linear combination of modified Bessel functions $I_\nu(z)$ and $K_\nu(\nu)$ [49]. By assuming that the initial magnetic power spectrum is such that $n_B < 7/2$ the solution of Eq. (5.17) is:

$$\mathcal{G}(k) = \frac{2^{1-\nu} A_B}{\Gamma(\nu)} \left(\frac{k}{k_L} \right)^{5/2} \left(\frac{k_\gamma}{k_L} \right)^{-\nu} K_\nu(k/k_\gamma), \quad k_\gamma = \sqrt{\frac{3\gamma}{2\lambda}}, \quad \nu = \frac{7}{2} - n_B, \quad (5.18)$$

which satisfies the correct boundary conditions since, in the limit $k \ll k_\gamma$, we have that $\mathcal{G}(k) \rightarrow A_B(k/k_L)^{n_B-1}$. The results obtained so far partially challenges the standard argument leading to the calculation of the magnetic diffusivity scale which is phenomenologically

important insofar as it determines the scale of the exponential suppression of the magnetic power spectrum. Prior to decoupling the magnetic diffusivity scale is estimated by requiring that $k_\sigma^2 \simeq 4\pi\sigma\mathcal{H}$. The latter condition stems directly from the magnetic diffusivity equation and it totally neglects the flow.

The solution (5.13) can be used to compute the evolution of the spectral index induced by the predecoupling flow. The spectral index of the magnetic field is defined as

$$N - 1 = \frac{\partial \ln \mathcal{P}_B}{\partial \ln k} \equiv \frac{1}{\mathcal{P}_B} \frac{\partial \mathcal{P}_B}{\partial z}. \quad (5.19)$$

Having assigned the spectrum at a fiducial (pivot) scale, it is interesting to ask what happens to the power spectrum for larger length-scales (i.e. smaller wavenumbers). From Eq. (5.13) it is possible to derive an evolution equation for N as it is defined in Eq. (5.19). After simple algebra the result is given by:

$$\frac{\partial N}{\partial \tau} = \gamma \left[-15 \frac{\partial N}{\partial z} + 3 \frac{\partial^2 N}{\partial z^2} + 6N \left(\frac{\partial N}{\partial z} \right) \right]. \quad (5.20)$$

The solution of Eq. (5.20) can be found in the form $N(z, \tau) = n_B - 1 + \delta N(z, \tau)$ with initial conditions $\delta N(z, 0) = 0$. In this case Eq. (5.20) becomes:

$$\frac{1}{\gamma} \frac{\partial \delta N}{\partial \tau} = 3 \frac{\partial^2 \delta N}{\partial z^2} + (6n_B - 21) \frac{\partial \delta N}{\partial z}. \quad (5.21)$$

Using the standard Laplace transform technique we have that

$$N(z, \tau) = n_B + \frac{|z|}{12\pi\gamma^3\tau^3} e^{-y^2(z, \tau)}, \quad y(z, \tau) = \frac{1}{\sqrt{12\gamma\tau}} \left[z + 6\gamma\tau \left(n_B - \frac{7}{2} \right) \right]^2. \quad (5.22)$$

The result of Eq. (5.22) matches pretty well with the result obtained directly from Eq. (5.19) by using the exact solution of Eq. (5.13), as it can be seen by plotting the respective functions for specific values of the spectral index. To avoid lengthy digression this analysis will be omitted.

5.2 Evolution in physical space

The Schrödinger-like equation in the compressible case (see Eqs. (4.30), (4.31) and (4.32)) allows for a semi-quantitative description of the limits $r \rightarrow 0$ (where magnetic diffusivity dominates) and $r \rightarrow \infty$ (where the large-scale flow dominates). To deduce the potential in the limit $r \rightarrow 0$ the correlation functions of the velocity can be expanded in the limit $R = rk_* < 1$ where k_* is an auxiliary scale in the space of the wavenumbers. Let us therefore start with $V_T(r)$ and $V_L(r)$ written as

$$\begin{aligned} V_T(r) &= \overline{\mathcal{M}}_{\mathcal{R}}^2(\tau_*) \mathcal{A}_{\mathcal{R}} \left(\frac{k_0}{k_p} \right)^{n_s-1} \int_{k_0/k_p}^{\infty} \frac{dx}{x} x^{n-1} \sin^2(x\alpha) e^{-2x^2\beta} A(x, R), \\ V_L(r) &= \overline{\mathcal{M}}_{\mathcal{R}}^2(\tau_*) \mathcal{A}_{\mathcal{R}} \left(\frac{k_0}{k_p} \right)^{n_s-1} \int_{k_0/k_p}^{\infty} \frac{dx}{x} x^{n-1} \sin^2(x\alpha) e^{-2x^2\beta} B(x, R), \end{aligned} \quad (5.23)$$

where k_0 can be estimated from the inverse of the comoving angular diameter distance to last scattering. In Eq. (5.23) the functions $A(x, R)$ and $B(x, R)$ are defined as

$$\begin{aligned} A(x, R) &= \left[\frac{1}{x^3 R^3} \sin xR - \frac{1}{x^2 R^2} \cos xR \right], \\ B(x, R) &= \left[-\frac{2}{x^3 R^3} \sin xR + \frac{2}{x^2 R^2} \cos xR + \frac{\sin xR}{xR} \right]. \end{aligned} \quad (5.24)$$

In Eq. (5.23) the following rescaled quantities have been introduced:

$$\alpha = r_s(\tau_*)k_*, \quad \beta = \frac{k_*^2}{k_d^2}, \quad x = \frac{k}{k_*}, \quad R = rk_*. \quad (5.25)$$

The expansion of $V_T(R)$ and $V_L(R)$ for $R < 1$ becomes therefore

$$V_T(R) = \mathcal{C}(n_s, \mathcal{A}_R) \left[\mathcal{I}(n_s, \alpha, \beta) - \mathcal{I}(n_s + 2, \alpha, \beta) \frac{R^2}{10} + \mathcal{I}(n_s + 4, \alpha, \beta) \frac{R^4}{280} + \dots \right] \quad (5.26)$$

$$V_L(R) = \mathcal{C}(n_s, \mathcal{A}_R) \left[\mathcal{I}(n_s, \alpha, \beta) - \mathcal{I}(n_s + 2, \alpha, \beta) \frac{3R^2}{10} + \mathcal{I}(n_s + 4, \alpha, \beta) \frac{R^4}{56} + \dots \right] \quad (5.27)$$

where

$$\mathcal{C}(n_s, \mathcal{A}_R) = \tau_c \frac{\mathcal{A}_R}{25} \left(\frac{k_*}{k_p} \right)^{n-1}, \quad \mathcal{I}(n_s, \alpha, \beta) = \int_0^\infty \frac{dx}{x} x^{n_s-1} \sin^2(x\alpha) e^{-2x^2\beta}. \quad (5.28)$$

To discuss in a semi-quantitative manner the properties of Eq. (4.30) it is useful (even if not necessary) to identify $k_* \simeq \pi/d_A(\tau_*)$. Following the approach of Zeldovich (see, e.g. [3]) it is useful to introduce an interpolating form of the correlation function of the velocity which is suppressed for $r \rightarrow \infty$ and leads to the correct limits of Eqs. (5.26) and (5.27) for $r \rightarrow 0$. The interpolating form is given by

$$V_L(r) = v_0 e^{-3\mu r^2}, \quad V_T(r) = v_0 e^{-\mu r^2}, \quad (5.29)$$

where v_0 has been already introduced in Eq. (3.14) and coincides with the normalization which can be derived from Eqs. (5.26)–(5.28) in the limit $\overline{\mathcal{M}}_R(\tau_*) \rightarrow \sqrt{3}/5$. Using Eq. (5.29) the effective potential of Eqs. (4.31)–(4.32) can be computed. For instance $g(r) = v_0[1 + \varepsilon - \exp(-3\mu r^2)]$ where $\varepsilon = \lambda/v_0 \ll 1$. In the limits $r \rightarrow 0$ and $r \rightarrow \infty$ the potential $W(r)$ becomes

$$\lim_{r\sqrt{\mu} \rightarrow 0} W(r) = \frac{2\lambda}{r^2}, \quad \lim_{r\sqrt{\mu} > 1} W(r) = \frac{2\overline{\lambda}}{r^2}, \quad (5.30)$$

showing that for intermediate and large scales the suppression of the correlation function of the magnetic field is determined by the large-scale flow contained in $\overline{\lambda}$. If $v_0 \gg \lambda$ the potential can become negative. Equation (4.30) may have negative “energy” levels, i.e. solutions for $\Psi(r, \tau) \simeq \exp[-2E\tau]\Phi(r)$ which are exponentially increasing in time (i.e. $E < 0$). For at least one level to exist we should have, roughly, that

$$\int_{r_1}^{r_2} \frac{\sqrt{E - W(r)}}{\sqrt{g(r)}} dr \geq \frac{\pi}{2}, \quad (5.31)$$

where r_1 and r_2 are the inversion points. Equation (5.31) can be verified but the growth rate of the field is $\Gamma \sim \mu v_0$ leads to a negligible integrated growth at last scattering, i.e. $\Gamma \tau_* \sim \mathcal{O}(10^{-14})$ even assuming $\sqrt{\mu} \sim 1/d_A(\tau_*)$. This semi-quantitative argument (also employed in the case of gyrotropic turbulence [3]) follows from the analogy with the Schrödinger equation in the WKB approximation.

6 Concluding remarks

The standard treatment adopted for the evolution of predecoupling magnetic fields usually neglects two aspects which are the starting point of the present investigation, namely the compressibility of the plasma and the large-scale flow induced by curvature perturbations. A third related coincidence is that, prior to decoupling, the magnetic Reynolds number is roughly 20 orders of magnitude larger than the kinetic Reynolds number which is, in turn, smaller than one. The basic idea has been to derive an integrodifferential equation valid for the compressible mode and for a standard adiabatic closure. In Fourier space, the diffusive approximation leads to a nonlocal diffusion equation similar to the ones often discussed in wave turbulence. In physical space the integrodifferential equation can be transformed into a Schrödinger-like equation whose effective potential depends on the spectral properties of the large-scale flow. Some applications have been discussed by solving the corresponding equations: they range from the effective evolution of the magnetic spectral index to the qualitative discussion of the spectrum of the eigenvalues of the equation for the magnetic power spectra in physical space.

In the absence of large-scale flow the magnetic power spectra are suppressed at small scales (i.e. large wavenumbers) because of the finite value of the conductivity which implies an effective (comoving) wavenumber k_σ

$$k_\sigma \simeq 2.5 \times 10^{10} \left(\frac{d_A}{14116 \text{ Mpc}} \right)^{-1/2} \text{ Mpc}^{-1}, \quad (6.1)$$

where $d_A \simeq 14116 \text{ Mpc}$ denotes the (comoving) angular diameter distance to last scattering. In units of d_A we have that $k_\sigma d_A \simeq \mathcal{O}(10^{14})$ implying that the corresponding length-scale is much smaller than the Hubble radius at last scattering, as expected. The results of the present paper suggest, however, that the correct magnetic diffusivity length-scale is much larger than k_σ^{-1} . Using the Markovian approximation it has been shown that the magnetic field can be averaged over the large-scale flow and the resulting diffusive scale can be estimated from Eq. (3.15) and it is

$$k_M \simeq 1.42 \times 10^3 \left(\frac{\mathcal{A}_R}{2.43 \times 10^{-9}} \right)^{-1/2} \text{ Mpc}^{-1}, \quad (6.2)$$

where the fiducial set of parameters of Eq. (2.4) has been assumed in the context of the vanilla Λ CDM scenario. Not only $k_M \ll k_\sigma$ but, in units of d_A , it turns out that $k_M d_A \simeq$

$\mathcal{O}(10^7)d_A$. From the analysis of the evolution equation of the power spectrum in the diffusive approximation it is possible to derive yet another dissipative scale, i.e. k_γ (see Eq. (5.18) for a definition) whose explicit value, always in terms of our fiducial set of parameters, is:

$$k_\gamma \simeq 2.16 \times 10^6 \left(\frac{\mathcal{A}_\mathcal{R}}{2.43 \times 10^{-9}} \right)^{1/2} \text{Mpc}^{-1}. \quad (6.3)$$

By comparing Eqs. (6.1), (6.2) and (6.3) it is clear that the following (approximate) hierarchy holds, i.e. $k_M < k_\gamma < k_\sigma$. Both the results for k_M and k_γ have been derived, directly or indirectly, by assuming the Markovian approximation for the velocity correlator. If the Markovian approximation is relaxed the diffusive wavenumber does not get larger but even smaller. From Eq. (B.22) it is possible to obtain that

$$k_{nM} \simeq 7.5 \times 10^{-2} \left(\frac{\mathcal{A}_\mathcal{R}}{2.43 \times 10^{-9}} \right)^{1/2} \text{Mpc}^{-1}. \quad (6.4)$$

In summary, it has been shown that the large-scale flow can affect the evolution of the magnetic power spectra at large scales not only by potentially shifting the effective spectral index but also by changing the diffusive scales. Thanks to the results derived in the present paper, the evolution of the magnetic power spectra prior to decoupling can be addressed in terms of a novel set of equations which take into account the effects of the large-scale flow and which can be explicitly solved in various physical limits. According to the present results it does not seem correct to treat the predecoupling plasma by simply assuming an incompressible flow; the latter closure is sound in the absence of large-scale curvature perturbations, in flat space-time and for high (kinetic and magnetic) Reynolds numbers. None of these three assumptions are verified after electron-positron annihilation and prior to last scattering. A closer scrutiny of the description developed here seems therefore both motivated and potentially rewarding.

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A Basic conventions

In this appendix some basic conventions will be summarized. As an example, the Fourier transform of the magnetic field and of the velocity field are defined as

$$B_i(\vec{x}, \tau) = \int \frac{d^3q}{(2\pi)^{3/2}} B_i(\vec{q}, \tau) e^{-i\vec{q}\cdot\vec{x}}, \quad v_i(\vec{x}, \tau) = \int \frac{d^3q}{(2\pi)^{3/2}} v_i(\vec{q}, \tau) e^{-i\vec{q}\cdot\vec{x}}. \quad (\text{A.1})$$

While the magnetic field \vec{B} is strictly divergenceless (i.e. $\vec{\nabla} \cdot \vec{B} = 0$), the velocity field may be either solenoidal (i.e. $\vec{\nabla} \cdot \vec{v} = 0$) or not solenoidal (i.e. $\vec{\nabla} \cdot \vec{v} \neq 0$). If the flow is strictly solenoidal it is also incompressible. In the vanilla Λ CDM case the flow is irrotational and compressible. The following definitions will also be employed in section 2:

$$\Omega_B(\vec{x}, \tau) = \int \frac{d^3k}{(2\pi)^{3/2}} \Omega_B(\vec{k}, \tau) e^{-i\vec{k}\cdot\vec{x}}, \quad \Pi_{ij}(\vec{x}, \tau) = \int \frac{d^3k}{(2\pi)^{3/2}} \Pi_{ij}(\vec{k}, \tau) e^{-i\vec{k}\cdot\vec{x}}, \quad (\text{A.2})$$

where $\Omega_B(\vec{k}, \tau)$ and $\Pi_{ij}(\vec{k}, \tau)$ are given by:

$$\Omega_B(\vec{k}, \tau) = \frac{1}{8\pi a^4 \rho_\gamma} \int \frac{d^3q}{(2\pi)^{3/2}} B_k(\vec{q}, \tau) B^k(\vec{k} - \vec{q}, \tau), \quad (\text{A.3})$$

$$\Pi_{ij}(\vec{k}, \tau) = \frac{1}{4\pi a^4} \int \frac{d^3q}{(2\pi)^{3/2}} \left[B_i(\vec{q}, \tau) B_j(\vec{k} - \vec{q}, \tau) - \frac{1}{3} B_k(\vec{q}, \tau) B^k(\vec{k} - \vec{q}, \tau) \delta_{ij} \right]. \quad (\text{A.4})$$

Recalling that $(p_\gamma + \rho_\gamma) \nabla^2 \sigma_B = \partial_i \partial_j \Pi_B^{ij}$ we also have, in Fourier space, that

$$\sigma_B(\vec{k}, \tau) = \frac{3 \hat{k}_i \hat{k}_j}{16\pi a^4 \rho_\gamma} \int \frac{d^3q}{(2\pi)^{3/2}} \left[B^i(\vec{q}, \tau) B^j(\vec{k} - \vec{q}, \tau) - \frac{\delta^{ij}}{3} B_k(\vec{q}, \tau) B^k(\vec{k} - \vec{q}, \tau) \right]. \quad (\text{A.5})$$

Finally Eqs. (A.2) and (A.5) imply

$$\sigma_B(\vec{k}, \tau) + \frac{1}{2} \Omega_B(\vec{k}, \tau) = \frac{3 \hat{k}_i \hat{k}_j}{16\pi a^4 \rho_\gamma} \int \frac{d^3q}{(2\pi)^{3/2}} B^i(\vec{q}, \tau) B^j(\vec{k} - \vec{q}, \tau). \quad (\text{A.6})$$

B Non Markovian approach

In the non-Markovian approximation the correlator of the velocities in the compressible case is still Gaussian and it is proportional to the correlator of the curvature perturbations:

$$\langle v_i(\vec{q}, \tau_1) v_j(\vec{p}, \tau_2) \rangle = \frac{q_i q_j}{q^2} \Gamma(q, \tau_1, \tau_2) \langle \mathcal{R}_*(\vec{q}) \mathcal{R}_*(\vec{p}) \rangle, \quad (\text{B.1})$$

having introduced the function $\Gamma(q, \tau_1, \tau_2)$

$$\Gamma(q, \tau_1, \tau_2) = \overline{\mathcal{M}}_{\mathcal{R}}(\tau_1) \overline{\mathcal{M}}_{\mathcal{R}}(\tau_2) \sin(q c_{\text{sb}} \tau_1) \sin(q c_{\text{sb}} \tau_2) e^{-q^2 \nu_{\text{th}}(\tau_1 + \tau_2)}, \quad (\text{B.2})$$

where $\overline{\mathcal{M}}_{\mathcal{R}}(\tau)$ is given by

$$\overline{\mathcal{M}}_{\mathcal{R}}(\tau) = \frac{1 + 3R_b}{\sqrt{3}(1 + R_b)^{3/4}} \mathcal{T}_{\mathcal{R}}(\tau) \rightarrow \frac{\sqrt{3}}{5}. \quad (\text{B.3})$$

The limit in the second expression holds after matter-radiation equality and for $R_b \rightarrow 0$. As explained this approximation is justified at last scattering where $R_b \sim 0.6$. The expression of Eq. (B.2) can also be written as:

$$\Gamma(q, \tau_1, \tau_2) = \frac{1}{2} \left\{ \cos[qc_{\text{sb}}(\tau_1 - \tau_2)] - \cos[qc_{\text{sb}}(\tau_1 + \tau_2)] \right\} e^{-q^2 \nu_{\text{th}}(\tau_1 + \tau_2)} \overline{\mathcal{M}}_{\mathcal{R}}(\tau_1) \overline{\mathcal{M}}_{\mathcal{R}}(\tau_2). \quad (\text{B.4})$$

With these notations the correlator of the velocities can be written as:

$$\langle v_i(\vec{q}, \tau_1) v_j(\vec{p}, \tau_2) \rangle = \frac{q_i q_j}{q^2} v(q) \Gamma(q, \tau_1, \tau_2) \delta^{(3)}(\vec{q} + \vec{p}), \quad v(q) = \frac{2\pi^2}{q^3} \mathcal{P}_{\mathcal{R}}(q). \quad (\text{B.5})$$

To simplify the problem it is practical to use the limit $\sigma \rightarrow \infty$ where the magnetic diffusivity equation is given by:

$$\frac{\partial \vec{B}}{\partial \tau} = \vec{\nabla} \times (\vec{v}_{\gamma b} \times \vec{B}). \quad (\text{B.6})$$

By writing Eq. (B.6) in Fourier space we have:

$$\partial_{\tau} B_i = \frac{(-i)}{(2\pi)^{3/2}} \int d^3 q \int d^3 p \delta^{(3)}(\vec{k} - \vec{q} - \vec{p}) \epsilon_{m n i} \epsilon_{a b n} (q_m + p_m) v_a(\vec{q}, \tau) B_b(\vec{p}, \tau), \quad (\text{B.7})$$

The solution of Eq. (B.7) can be formally written as:

$$B_i(\vec{k}, \tau) = \frac{(-i)}{(2\pi)^{3/2}} \int_0^{\tau} d\tau_1 \int d^3 q \int d^3 p \delta^{(3)}(\vec{k} - \vec{q} - \vec{p}) \epsilon_{m n i} \epsilon_{a b n} (q_m + p_m) v_a(\vec{q}, \tau_1) B_b(\vec{p}, \tau_1). \quad (\text{B.8})$$

Equation (B.8) can be solved by iteration as

$$\begin{aligned} B_i(\vec{k}, \tau) &= \sum_{n=0}^{\infty} B_i^{(n)}(\vec{k}, \tau), \\ B_i^{(n+1)}(\vec{k}, \tau) &= \frac{(-i)}{(2\pi)^{3/2}} \int_0^{\tau} d\tau_1 \int d^3 q \int d^3 p \delta^{(3)}(\vec{k} - \vec{q} - \vec{p}) \\ &\quad \times \epsilon_{m n i} \epsilon_{a b n} (q_m + p_m) v_a(\vec{q}, \tau_1) B_b^{(n)}(\vec{p}, \tau_1). \end{aligned} \quad (\text{B.9})$$

We can then average the magnetic field over the velocity field. The terms containing an odd number of velocities will be zero while the correlators containing an even number of velocities do not vanish, in formulas

$$\langle B_i^{(2n+1)} \rangle = H_i^{(2n+1)} = 0, \quad \langle B_i^{(2n+2)} \rangle = H_i^{(2n+2)} \neq 0. \quad (\text{B.10})$$

This conclusion holds both in the Markovian and in the non-Markovian case since it is directly related to the Gaussianity of the curvature perturbations. Let us now write explicitly the contributions up to second order and let us compute the first few terms in the expansion:

$$B_i^{(2)}(\vec{k}, \tau) = \frac{(-i)}{(2\pi)^{3/2}} \int d^3q \int d^3p \delta^{(3)}(\vec{k} - \vec{q} - \vec{p}) \times \epsilon_{mni} \epsilon_{abn} (q_m + p_m) \int_0^\tau d\tau_1 v_a(\vec{q}, \tau_1) B_b(\vec{p}, \tau_1), \quad (\text{B.11})$$

$$B_b^{(1)}(\vec{p}, \tau_1) = \frac{(-i)}{(2\pi)^{3/2}} \int d^3q' \int d^3p' \delta^{(3)}(\vec{p} - \vec{q}' - \vec{p}') \times \epsilon_{bm'n'} \epsilon_{a'b'n'} (q'_{m'} + p'_{m'}) \int_0^{\tau_1} d\tau_2 v_{a'}(\vec{q}', \tau_2) B_b(\vec{p}'), \quad (\text{B.12})$$

where we used that $B_b^{(0)}(\vec{p}', \tau_2) = B_b(\vec{p}')$. Inserting Eq. (B.12) inside Eq. (B.11) we have that

$$B_i^{(2)}(\vec{k}, \tau) = \frac{(-i)^2}{(2\pi)^3} \int d^3q \int d^3p \int d^3q' \int d^3p' \delta^{(3)}(\vec{k} - \vec{q} - \vec{p}) \delta^{(3)}(\vec{p} - \vec{q}' - \vec{p}') \times \int_0^\tau d\tau_1 \int_0^{\tau_1} d\tau_2 (q_m + p_m) (q'_{m'} + p'_{m'}) \epsilon_{bm'n'} \epsilon_{a'b'n'} \epsilon_{mni} \epsilon_{abn} \times v_{a'}(\vec{q}', \tau_2) v_a(\vec{q}, \tau_1) B_{b'}(\vec{p}'). \quad (\text{B.13})$$

By averaging over the velocity fields we have that:

$$H_i^{(2)}(\vec{k}, \tau) = \frac{(-i)^2}{(2\pi)^3} \int d^3q \int d^3p \int d^3q' \int d^3p' \delta^{(3)}(\vec{k} - \vec{q} - \vec{p}) \delta^{(3)}(\vec{p} - \vec{q}' - \vec{p}') \times \int_0^\tau d\tau_1 \int_0^{\tau_1} d\tau_2 (q_m + p_m) (q'_{m'} + p'_{m'}) \epsilon_{bm'n'} \epsilon_{a'b'n'} \epsilon_{mni} \epsilon_{abn} \times \langle v_{a'}(\vec{q}', \tau_2) v_a(\vec{q}, \tau_1) \rangle B_{b'}(\vec{p}'), \quad (\text{B.14})$$

where $\langle B_i^{(2)}(\vec{k}, \tau) \rangle = H_i^{(2)}(\vec{k}, \tau)$. After inserting the explicit expression of the correlator obtained in Eq. (B.5), (Eq. (B.14) reduces to

$$H_i^{(2)}(\vec{k}, \tau) = \frac{(-i)^2}{(2\pi)^3} \int d^3q \int d^3p \int d^3q' \int d^3p' \delta^{(3)}(\vec{k} - \vec{q} - \vec{p}) \delta^{(3)}(\vec{p} - \vec{q}' - \vec{p}') \times \int_0^\tau d\tau_1 \int_0^{\tau_1} d\tau_2 \Gamma(q, \tau_1, \tau_2) \frac{q_a q_{a'}}{q^2} v(q) \delta^{(3)}(\vec{q} + \vec{q}') \times (q_m + p_m) (q'_{m'} + p'_{m'}) \epsilon_{bm'n'} \epsilon_{a'b'n'} \epsilon_{mni} \epsilon_{abn} B_{b'}(\vec{p}'). \quad (\text{B.15})$$

Using the three delta functions over the momenta the expression of Eq. (B.15) becomes:

$$H_i^{(2)}(\vec{k}, \tau) = \frac{(-i)^2}{(2\pi)^3} \int d^3q v(q) B_{b'}(\vec{k}) \int_0^\tau d\tau_1 \int_0^{\tau_1} d\tau_2 \Gamma(q, \tau_1, \tau_2) \times \frac{q_a q_{a'}}{q^2} k_m (k_{m'} - q_{m'}) \epsilon_{bm'n'} \epsilon_{a'b'n'} \epsilon_{mni} \epsilon_{abn}. \quad (\text{B.16})$$

Eq. (B.16) can be also written, after some algebra, as

$$H_i^{(2)}(\vec{k}, \tau) = \frac{(-i)^2}{(2\pi)^3} \int d^3q v(q) B_i(\vec{k}) \frac{(\vec{k} \cdot \vec{q})[(\vec{k} \cdot \vec{q}) - q^2]}{q^2} \times \int_0^\tau d\tau_1 \int_0^{\tau_1} d\tau_2 \Gamma(q, \tau_1, \tau_2). \quad (\text{B.17})$$

By appreciating that $\Gamma(q, \tau_1, \tau_2) = \Gamma(q, \tau_2, \tau_1)$ the integrations over τ_1 and τ_2 can be explicitly performed by recalling the elementary integral

$$\int_0^y \sin(bx) e^{-ax} dx = \frac{b}{a^2 + b^2} - \frac{b}{a^2 + b^2} \left[\cos by + \frac{a}{b} \sin by \right] e^{-ay}. \quad (\text{B.18})$$

The result is:

$$\bar{\gamma}(q, \tau) = \int_0^\tau d\tau_1 \int_0^{\tau_1} d\tau_2 \Gamma(q, \tau_1, \tau_2) = \frac{\{1 - e^{-q^2 \nu_{\text{th}} \tau} [\cos(q c_{\text{sb}} \tau) + \epsilon(q, \tau) \sin(q c_{\text{sb}} \tau)]\}^2}{2q^2 c_{\text{sb}}^2 [1 + \epsilon^2(q, \tau)]^2}, \quad (\text{B.19})$$

where $\epsilon(q, \tau) = q^2 \nu_{\text{th}}^2 / c_{\text{sb}}^2 \ll 1$. In the limit $\epsilon(q, \tau) \ll 1$ and $q^2 \nu_{\text{th}} \tau \gamma(q\tau) \simeq [c_{\text{sb}}^2 \tau^4]/8$ and the higher order contributions

$$\begin{aligned} H_i^{(2)}(\vec{k}, \tau) &= -\frac{k^2 B_i(\vec{k})}{3} \int \frac{d^3q}{(2\pi)^3} \gamma(q, \tau) v(q), \\ H_i^{(4)}(\vec{k}, \tau) &= \frac{1}{2!} \frac{k^2 B_i(\vec{k})}{3} \int \frac{d^3q}{(2\pi)^3} \gamma(q, \tau) v(q) \int \frac{d^3p}{(2\pi)^3} \gamma(p, \tau) v(p), \\ H_i^{(6)}(\vec{k}, \tau) &= -\frac{1}{3!} \frac{k^2 B_i(\vec{k})}{3} \int \frac{d^3q}{(2\pi)^3} \gamma(q, \tau) v(q) \int \frac{d^3p}{(2\pi)^3} \gamma(p, \tau) v(p) \\ &\quad \times \int \frac{d^3k}{(2\pi)^3} \gamma(k, \tau) v(k), \end{aligned} \quad (\text{B.20})$$

and so on can be resummed leading to an average magnetic field

$$H_i(\vec{k}, \tau) = e^{-f(k, \tau)} B_i(\vec{k}), \quad f(k, \tau) = \frac{c_{\text{sb}}^2}{24(n+1)} \mathcal{A}_{\mathcal{R}} k_{\text{p}}^2 k^2 \tau^4 \left(\frac{k_{\text{d}}}{k_{\text{p}}} \right)^{n+1}. \quad (\text{B.21})$$

The initial magnetic field is suppressed when $f(k, \tau_*) \geq 1$, i.e. when

$$\frac{k_{\text{nM}}}{k_{\text{p}}} \geq 20 \frac{\sqrt{2(n+1)}}{c_{\text{sb}} (d_{\text{A}} k_{\text{p}})^2} \mathcal{A}_{\mathcal{R}}^{-1/2} \left(\frac{k_{\text{p}}}{k_{\text{d}}} \right)^{(n+1)/2}, \quad (\text{B.22})$$

where d_{A} is the (comoving) angular diameter distance to last scattering and k_{nM} denotes the diffusion scale in the non-Markovian approach proposed in this appendix. The results obtained here are consistent with a rough dimensional going, in short, as follows. Equation (B.6) does not possess stationary solutions (see, e.g. [21]): the field is either amplified or dissipated. The typical diffusivity scale induced by large-scale curvature perturbations can be simply obtained by balancing the left and the right-hand sides of Eq. (B.6) and by assuming that the typical amplitude of the velocity field is given by $v_{\gamma\text{b}} \sim \sqrt{\mathcal{A}_{\mathcal{R}}}$ (for $n_{\text{s}} \simeq 1$). In this case the typical diffusivity length is $L \sim \sqrt{\mathcal{A}_{\mathcal{R}}} \tau \sim \sqrt{\mathcal{A}_{\mathcal{R}}} \mathcal{H}^{-1}$.

References

- [1] H. K. Moffat, *Magnetic field generation in electrically conducting fluids*, (Cambridge University Press, Cambridge 1978).
- [2] E. Parker, *Cosmical Magnetic Fields* (Oxford University Press, Oxford, 1979).
- [3] Ya. B. Zeldovich, A. A. Ruzmaikin, and D. Sokoloff *Magnetic Fields in Astrophysics* (Gordon and Breach, New York 1983).
- [4] D. Biskamp, *Magnetohydrodynamic Turbulence*, (Cambridge University Press, 2003).
- [5] A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics*, (Dover Publications, Mineola, New York).
- [6] H. Kurki-Suonio, V. Muhonen, J. Valiviita, Phys. Rev. **D71**, 063005 (2005); R. Keski-talo, H. Kurki-Suonio, V. Muhonen, J. Valiviita, JCAP **0709**, 008 (2007).
- [7] K. Enqvist, H. Kurki-Suonio, J. Valiviita, Phys. Rev. **D62**, 103003 (2000); K. Enqvist, H. Kurki-Suonio, Phys. Rev. **D61**, 043002 (2000).
- [8] V. Rubakov and A. Vlasov, arXiv:1008.1704 [astro-ph.CO].
- [9] C. L. Bennett *et al.*, Astrophys. J. Suppl. **192**, 17 (2011); N. Jarosik *et al.*, Astrophys. J. Suppl. **192**, 14 (2011).
- [10] J. L. Weiland *et al.*, Astrophys. J. Suppl. **192**, 19 (2011); D. Larson *et al.*, Astrophys. J. Suppl. **192**, 16 (2011).
- [11] B. Gold *et al.*, Astrophys. J. Suppl. **192**, 15 (2011); E. Komatsu *et al.*, Astrophys. J. Suppl. **192**, 18 (2011).
- [12] M. Giovannini, Phys. Lett. **B668**, 44-50 (2008); Class. Quant. Grav. **26**, 045004 (2009).
- [13] W. Zhao, D. Baskaran, Phys. Rev. **D79**, 083003 (2009); W. Zhao, W. Zhang, Phys. Lett. **B677**, 16 (2009).
- [14] M. Giovannini, Phys. Rev. **D84**, 063010 (2011); Phys. Rev. **D79**, 103007 (2009); Phys. Rev. **D79**, 121302 (2009); M. Giovannini and N. Q. Lan, Phys. Rev. D **80**, 027302 (2009).
- [15] C.-P. Ma and E. Bertschinger, *Astrophys. J.* **455**, 7 (1995); J. Bardeen, P. Steinhardt, M. Turner, *Phys. Rev. D* **28**, 679 (1983).
- [16] M. Giovannini, Phys. Rev. D **73**, 101302 (2006); Phys. Rev. D **74**, 063002 (2006); PMC Phys. A **1**, 5 (2007).

- [17] A. P. Kazantsev, Sov. Phys. JETP **26**, 1031 (1968) [Zh. Eksp. teor. Fiz. **53**, 1806 (1967)].
- [18] R. H. Kraichnan and S. Nagarajan, Phys. Fluids **10**, 859 (1967).
- [19] R. H. Kraichnan, Phys. Rev. **107**, 1485 (1957); Phys. Rev. **109**, 1407 (1958).
- [20] S. Nagarajan, Astrophys. J. **134**, 447 (1961).
- [21] S. I. Vainshtein, Sov. Phys. Doklady **15**, 1090 (1971) [Dokl. Akad. Nauk SSSR **195**, 793 (1970)]; Sov. Phys. JETP **31**, 87 (1970) [Zh. Eksp. Teor. Fiz. **58**, 153 (1970)]; Sov. Phys. JETP **34**, 327 (1971) [Zh. Eksp. Teor. Fiz. **61**, 612 (1971)].
- [22] E. N. Parker, Astrophys. J. **157**, 1119 (1969).
- [23] R. Brandenberger, R. Kahn, and W. Press, *Phys. Rev. D* **28**, 1809 (1983); D. H. Lyth, *Phys. Rev. D* **31**, 1792 (1985).
- [24] M. Giovannini, *A primer on the Physics of the Cosmic Microwave Background*, (World Scientific, Singapore, 2008), p. 362.
- [25] L. Spitzer, *Physics of Fully ionized plasmas* (J. Wiley and Sons, New York, 1962).
- [26] T. Elperin, N. Kleeorin, and I. Rogachevskii, Phys. Rev. E **53**, 3431 (1996).
- [27] H. K. Moffat, Rep. Prog. Phys. **46**, 621 (1983); J. Fluid Mech. **106**, 27 (1981).
- [28] N. Kleeorin and I. Rogachevskii, Phys. Rev. E **50**, 493 (1994).
- [29] N. A. Krall, A. W. Trivelpiece: *Principles of Plasma Physics*, (San Francisco Press, San Francisco 1986).
- [30] J. Silk, Astrophys. J. **151**, 459 (1968).
- [31] P. Olesen, Phys. Lett. B **398**, 321 (1997); P. Olesen, NATO ASI Series B **366**, 159 (1998).
- [32] R. Banerjee and K. Jedamzik, Phys. Rev. D **70**, 123003 (2004).
- [33] M. Giovannini, CERN-PH-TH/2011-281, arXiv:1111.3867 [astro-ph.CO].
- [34] C. F. Von Weizsäcker, Astrophys. J. **114**, 165 (1951); G. Gamow, Phys. Rev. **86**, 251 (1952).
- [35] J. H. Oort, Nature **224**, 1158 (1969); L. M. Ozernoy and A. D. Chernin, Soviet Astron. AJ **12**, 901 (1969) [Astron. Zh. **45**, 1137 (1968)]; L. M. Ozernoy, Soviet Astron. AJ **15**, 923 (1972) [Astron. Zh. **48**, 1160 (1971)].
- [36] P. J. E. Peebles, Astrophys. and Space Sci. **11**, 443 (1971).

- [37] J. D. Barrow, Mon. Not. R. astr. Soc. **179**, 47 (1977); Mon. Not. R. astr. Soc. **178**, 625 (1977).
- [38] C. Hogan, Phys. Rev. Lett. **51** 1488 (1983); M. Hindmarsh, A. Everett, Phys. Rev. **D58**, 103505 (1998).
- [39] A. Brandenburg, K. Enqvist and P. Olesen Phys. Rev. D **54**, 1291 (1996); A. Brandenburg, K. Enqvist and P. Olesen Phys. Lett. B **392** , 395 (1997).
- [40] K. Enqvist, Int. J. Mod. Phys. D **7**, 331 (1998); D. T. Son, Phys. Rev. D **59**, 063008 (1999); M. Christensson, M. Hindmarsh, Phys. Rev. **D60**, 063001 (1999).
- [41] P. J. E. Peebles and J. T. Yu, Astrophys. J. **162** 815 (1970).
- [42] H. Jorgensen, E. Kotok, P. Naselsky, and I Novikov, Astron. Astrophys. **294**, 639 (1995); P. Naselsky and I. Novikov, Astrophys. J. **413**, 14 (1993).
- [43] U. Seljak, Astrophys. J. **435**, L87 (1994).
- [44] W. Hu and N. Sugiyama, Astrophys. J. **444**, 489 (1995).
- [45] M. Giovannini, Class. Quantum Grav. **27**, 105011 (2010).
- [46] V. E. Zakharov and R. Z. Sagdeev, Sov. Phys. Dokl. **15**, 439 (1970) [Dokl. Akad. Nauk. **192**, 297 (1970)].
- [47] P. H. Diamond, S.-I. Itoh, and K. Itoh, *Modern Plasma Physics*, (Cambridge University Press, Cambridge, 2010).
- [48] S. I. Vainshtein and L. L. Kichatinov, J. Fluid. Mech. **168**, 73 (1986).
- [49] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1972).